Black Box Work Extraction and Composite Hypothesis Testing

Kaito Watanabe
University of Tokyo

Joint work with Ryuji Takagi

Phys. Rev. Lett. 133, 250401(2024)

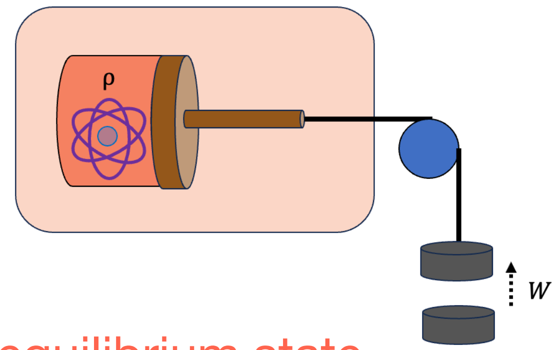
Quantum Thermodynamics

[Horodecki, Oppenheim, Nat. Commun.(2019)],[Faist, Renner, PRX (2018)]

Consider the states in a thermal bath of fixed temperature $T = \dfrac{1}{\beta}$

Quantum thermodynamics=Resource theory of **nonequilibriumness**

Free state: $\mathbb{F} = \{\tau\}, \quad \tau = \dfrac{e^{-\beta H}}{\mathrm{Tr}[e^{-\beta H}]}$ (Gibbs thermal state)



Resource state=Nonequilibrium state

τ

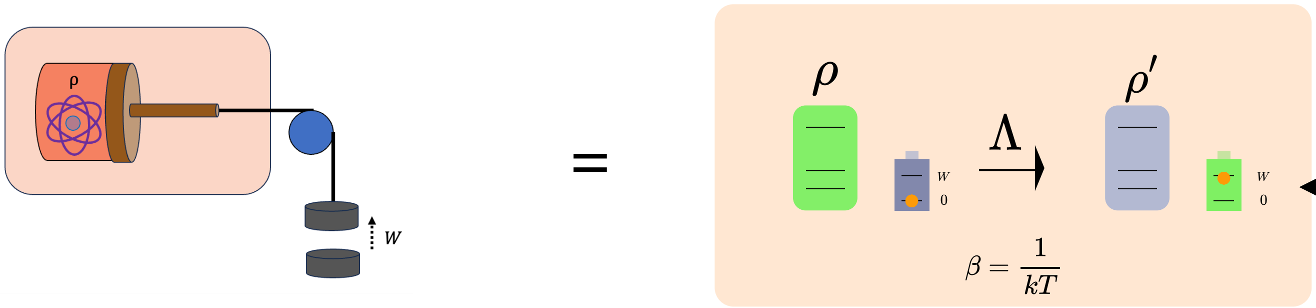
$\mathbb{O}$:free operation

$\mathbb{F}$: thermal state

Necessary condition for free operations $\mathbb{O}$: $\Lambda(\tau) = \tau, \quad \forall \Lambda \in \mathbb{O}$

Work extraction

[Horodecki, Oppenheim, Nat. Commun. (2013)]
[Brandão et al., PRL (2013)]



Work battery
$\mathscr{H}_W = \mathrm{Span}\{\,|0\rangle, |W\rangle\}$
$H_W = 0|0\rangle\langle 0| + W|W\rangle\langle W|$

- **One-shot optimal extractable work of state ρ with free operation $\mathbb{O}$**

$$\boxed{W^{\varepsilon}_{\mathbb{O}}(\rho) = \max\{W \mid \exists \Lambda \in \mathbb{O} \text{ s.t. } F(\Lambda(\rho), |W\rangle\langle W|) \geq 1-\varepsilon\}}$$ $\quad (F(\rho,\tau) = \|\sqrt{\rho}\sqrt{\tau}\|_1^2)$

- **Thermodynamic limit = Asymptotic optimal extractable work rate**

$$\boxed{W^{\mathrm{asymp}}_{\mathbb{O}}(\rho) = \lim_{\varepsilon\to+0}\limsup_{n\to\infty}\frac{1}{n}W^{\varepsilon}_{\mathbb{O}}(\rho^{\otimes n})}$$

Work extraction

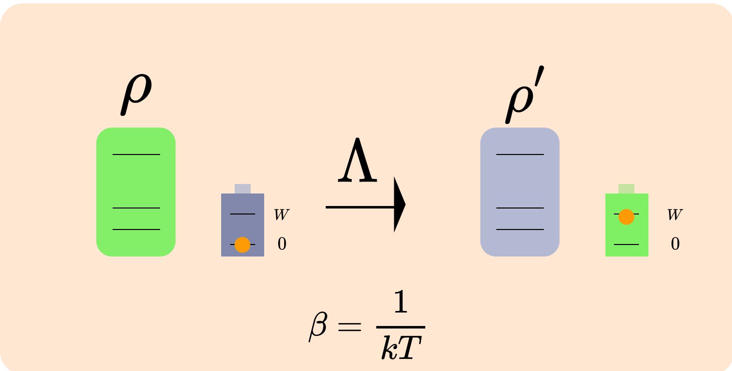
free operations $\mathbb{O}$ = Gibbs preserving operation $\quad \Lambda(\tau) = \tau, \quad \forall \Lambda \in \mathbb{O}$

$$W^{\varepsilon}_{\mathrm{GPO}}(\rho) = k_B T D^{\varepsilon}_{H}(\rho \| \tau)$$

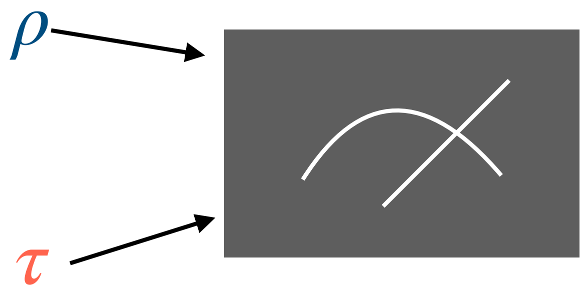
Hypothesis testing divergence

[Buscemi et al., Quantum(2019)]
[Wang, Wilde, PRL(2019)]
[Gour, PRX Quantum (2022)]

The task of discriminating between two states



Correspondence



Hypothesis testing

The task of distinguishing between two states

Type I error: guess ρ as τ (prob. $\mathrm{Tr}[\rho(I-M)]$)

Type II error: guess τ as ρ (prob. $\mathrm{Tr}[\tau M]$)

Hypothesis testing divergence

$$\boxed{D_H^{\varepsilon}(\rho\|\tau) := -\log \inf_{\substack{0 \le M \le I \\ \mathrm{Tr}[\rho(I-M)] \le \varepsilon}} \mathrm{Tr}[\tau M]}$$

Quantum Stein's Lemma

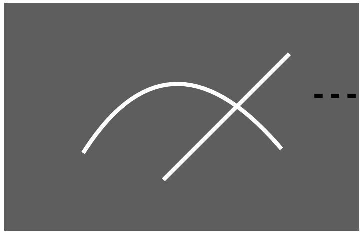
Asymptotic behavior of hypothesis testing divergence

$$D_H^{\varepsilon}(\rho\,||\,\tau) := -\log \inf_{\substack{0 \le M \le I \\ \mathrm{Tr}[\rho(I-M)] \le \varepsilon}} \mathrm{Tr}[\tau M]$$

Null hypothesis $\rho^{\otimes n}$

Alternative hypothesis $\tau^{\otimes n}$

$\{\overset{\rho}{M}, \overset{\tau}{I - M}\}$



Guess which is actually given

Quantum Stein's lemma

$$\boxed{\lim_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n}\|\tau^{\otimes n}) = D(\rho\|\tau) \quad \forall \varepsilon \in (0,1)}$$

[Hiai, Petz, Comm. Mat. Phys. (1991)]

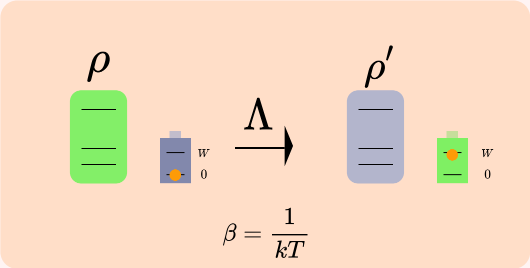
[Ogawa, Nagaoka, IEEE. Trans. Info. Theory., (2000)]

Umegaki relative entropy

$$D(\rho\,||\,\tau) := \mathrm{Tr}[\rho\log\rho - \rho\log\tau]$$

Thermodynamic limit

Work extraction



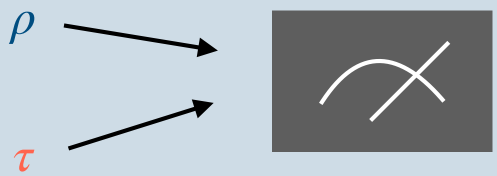
One-shot extractable work $W^{\varepsilon}_{\mathrm{GPO}}(\rho)$ $=$ $k_B T D^{\varepsilon}_{H}(\rho \| \tau)$ Hypothesis testing divergence

Thermodynamic limit

Quantum Stein's lemma

$W^{\mathrm{asymp}}_{\mathrm{GPO}}(\rho) = k_B T D(\rho \| \tau)$

Hypothesis testing



$$k_B T D(\rho \| \tau) = \underbrace{\mathrm{Tr}[\rho H] - T \cdot k_B \mathrm{Tr}[-\rho \log \rho]}_{F(\rho) = E - TS} - (\underbrace{-k_B T \log Z}_{F(\tau)})$$

Cost for the information about the initial state

The assumption in the previous setting for the work extraction

=The experimenters have **complete information** **about the initial state** ρ
to tailor the protocol depending on the initial state

But is this really possible?

- Unknown noise from the environment
- Huge cost for the state tomography
  - Copies of the initial state
  - The resource needed to apply the measurement

Questions

- How much work can one extract **without complete information** about the state?

Black box work extraction

The initial state is picked from a set $\mathscr{S} \subset \mathscr{D}(\mathscr{H})$ of states called a black box

- We know: the description of the black box $\mathscr{S}$
- We do not know: which state is picked up

Worst-case work extraction

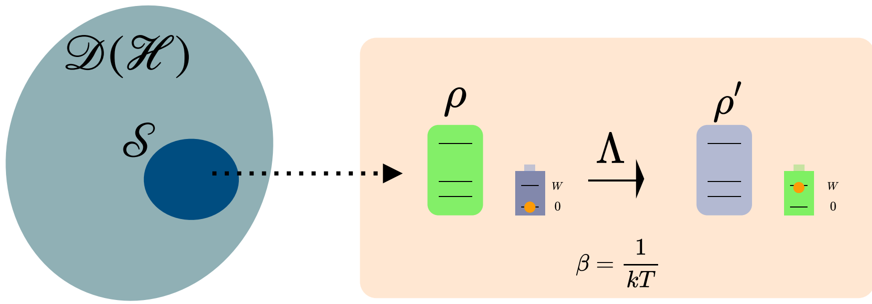
- **The one-shot extractable work from the black box $\mathscr{S}$**

$$\boxed{W^{\varepsilon}_{\mathbb{O}}(\mathscr{S}) = \max\{W \mid \exists \Lambda \in \mathbb{O} \text{ s.t. } \min_{\rho \in \mathscr{S}} F(\Lambda(\rho), |W\rangle\langle W|) \geq 1 - \varepsilon\}}$$

(cf.) $W^{\varepsilon}_{\mathbb{O}}(\rho) = \max\{W \mid \exists \Lambda \in \mathbb{O} \text{ s.t. } F(\Lambda(\rho), |W\rangle\langle W|) \geq 1 - \varepsilon\}$

Note:
The operation cannot depend on the initial state

Note:
The state is not picked following some prob. dist.



Black box work extraction by Gibbs-Preserving operations

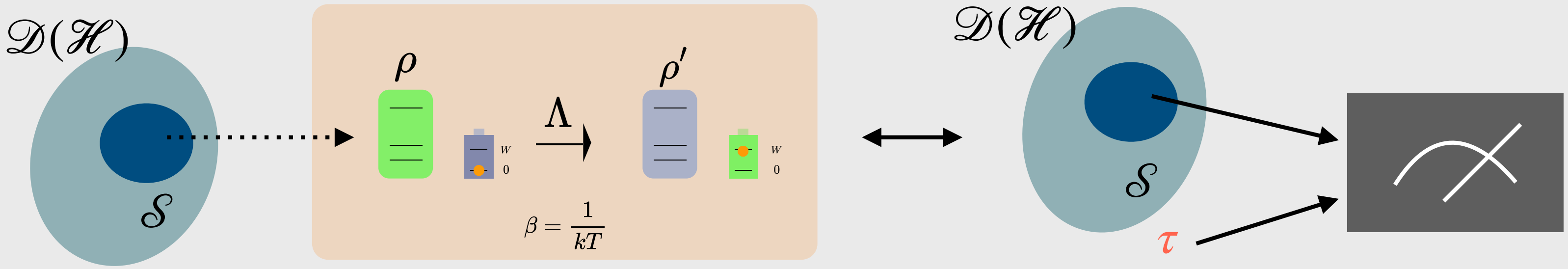
Result

$$\boxed{W^{\varepsilon}_{\mathrm{GPO}}(\mathcal{S}) = k_B T D^{\varepsilon}_{H}(\mathcal{S}\,||\,\tau)}$$

$$\Lambda(\tau_A) = \tau_B$$

Optimal guaranteed extractable work from black box $\mathcal{S}$

Distinguishability between $\mathcal{S}$ and thermal state τ



Composite hypothesis testing

Composite hypothesis testing divergence

$$D^{\varepsilon}_{H}(\mathcal{S}\,||\,\tau) = -\log \min_{\substack{0 \le M \le I \\ \sup_{\rho\in\mathcal{S}} \mathrm{Tr}[\rho(I-M)] \le \varepsilon}} \mathrm{Tr}[\tau M]$$

- Fully characterize the optimal extractable work via composite hypothesis testing
- Give an operational interpretation of the composite hypothesis testing divergence with composite 1st argument

Thermodynamic limit

Thermodynamic limit of black box work extraction

→The work extraction from the sequence of the black boxes $\{\mathcal{S}_n \subset \mathcal{D}(\mathcal{H}^{\otimes n})\}_{n\in\mathbb{N}}$

- **Asymptotic optimal extractable work rate from the sequence of black boxes**

$$\boxed{W_{\mathbb{O}}^{\mathrm{asymp}}(\{\mathcal{S}_n\}) := \lim_{\varepsilon\to+0}\limsup_{n\to\infty}\frac{1}{n}W_{\mathbb{O}}^{\varepsilon}(\mathcal{S}_n)}$$

$$W_{\mathrm{GPO}}^{\varepsilon}(\mathcal{S}) = k_B T D_H^{\varepsilon}(\mathcal{S}\,||\,\tau)$$

Result [Nötzel J. Phys. A, (2014)][Bergh et al. arxiv 2303.02016(2023)] [Bjelaković et al., Commun.Math. Phys.(2005)]

$$\mathcal{S}_n^{\mathrm{TP}} = \{\bigotimes_{i=1}^{n}\rho_i \mid \rho_i \in S \subset \mathcal{D}(\mathcal{H})\} \Rightarrow \boxed{W_{\mathrm{GPO}}^{\mathrm{asymp}}(\{\mathcal{S}_n^{\mathrm{TP}}\}_n) = k_B T \min_{\rho\in\mathscr{C}(S)} D(\rho\|\tau)}$$

$$\mathcal{S}_n^{\mathrm{IID}} = \{\rho^{\otimes n} \mid \rho \in S\} \Rightarrow \boxed{W_{\mathrm{GPO}}^{\mathrm{asymp}}(\{\mathcal{S}_n^{\mathrm{IID}}\}_n) = k_B T \min_{\rho\in S} D(\rho\|\tau)}$$

$\mathscr{C}(S)$: the convex hull of S

Thermodynamic limit

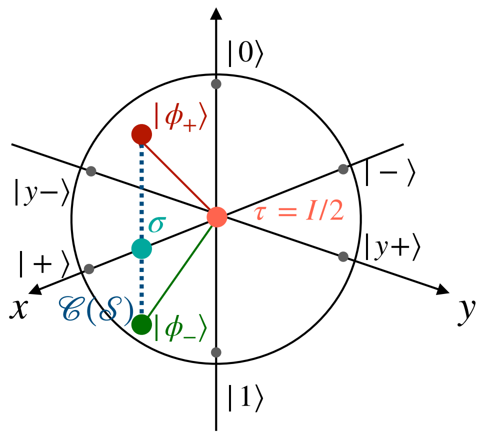
$$\mathcal{S}_n^{\mathrm{TP}} = \{\bigotimes_{i=1}^{n} \rho_i \mid \rho_i \in S \subset \mathcal{D}(\mathcal{H})\} \Rightarrow \boxed{W_{\mathrm{GPO}}^{\mathrm{asymp}}(\{\mathcal{S}_n^{\mathrm{TP}}\}_n) = k_B T \min_{\rho \in \mathcal{C}(S)} D(\rho \| \tau)}$$

E.g.) qubit state $H = EI_2$ $S = \{|\phi_+\rangle\langle\phi_+|, |\phi_-\rangle\langle\phi_-|\}$

$$\mathcal{S}_n = \{\bigotimes_{i=1}^{n} \rho_i \mid \rho_i \in \{|\phi_+\rangle, |\phi_-\rangle\}\}$$

$$|\phi_+\rangle = \frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle$$

$$|\phi_-\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$$



State-aware work extraction $\beta W_{\mathrm{GPO}}^{\mathrm{asymp}}(\{\otimes_i^n \rho_i\}_n) \geq 1$

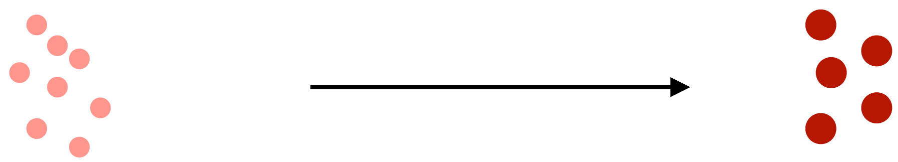
$$1 - h(1/50)$$
$$\|$$

Black box work extraction $\beta W_{\mathrm{GPO}}^{\mathrm{asymp}}(\{\mathcal{S}_n\}_n) = \min_{\rho \in \mathcal{C}(S)} D(\rho \| \tau) \leq D(\sigma \| \tau) < 1$

$\therefore W_{\mathrm{GPO}}^{\mathrm{asymp}}(\{\mathcal{S}_n\}_n) < \min_{\rho_n \in \mathcal{S}} W_{\mathrm{GPO}}^{\mathrm{asymp}}(\rho_n)$ in this example

Resource Distillation

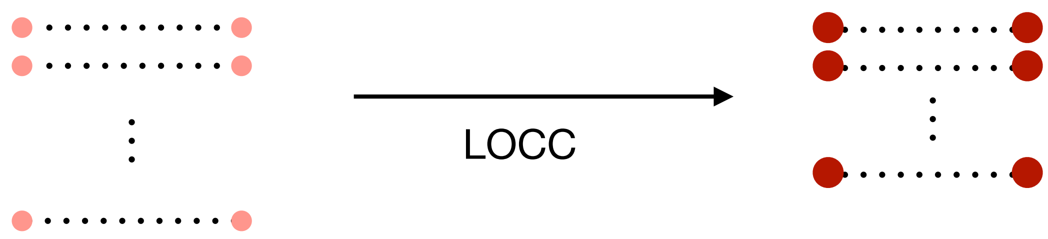
Obtain important resource states from noisy resource states



$\mathbb{O}$: free operation

$\rho^{\otimes n}$ $\longrightarrow$ $\Phi^{\otimes rn}$

E.g.) Entanglement distillation [Bennet et al., PRA(1995)]



LOCC

$|\phi\rangle^{\otimes n}$ $\longrightarrow$ $|\mathrm{Bell}\rangle^{\otimes rn}$

$r = S(|\phi\rangle)$

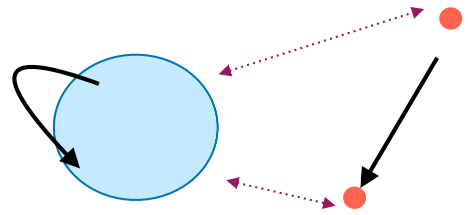
Entanglement entropy

- Teleportation
- Superdense coding

Black box resource distillation

The family of the target (pure) resource states $\mathbb{T} = \{\phi_i\}_{i\in\mathbb{N}}$

Free operations $\mathbb{O}_{\mathrm{RNG}} = \{\Lambda \mid \Lambda(\mathbb{F}) \subset \mathbb{F}\}$



The one-shot distillable resource from the black box $\mathcal{S}$

$$\boxed{d^{\varepsilon}_{\mathbb{O},R}(\mathcal{S}) = \max\{R(\phi_i),\ \phi_i \in \mathbb{T},\ \exists\Lambda \in \mathbb{O},\ \mathrm{s.t.}\ \min_{\rho\in\mathcal{S}} F(\Lambda(\rho),\phi_i) \geq 1-\varepsilon\}}$$

(Cf. $(d^{\varepsilon}_{\mathbb{O},R}(\rho) = \max\{R(\phi_i),\ \phi_i \in \mathbb{T},\ \exists\Lambda \in \mathbb{O},\ \mathrm{s.t.}\ F(\Lambda(\rho),\phi_i) \geq 1-\varepsilon\})$)

Result

$$\boxed{d^{\varepsilon}_{\mathbb{O}_{\mathrm{RNG}},R_s}(\mathcal{S}) = \lfloor D^{\varepsilon}_{H}(\mathcal{S}\,||\,\mathbb{F})\rfloor_{\mathbb{T}} \quad \text{or} \quad d^{\varepsilon}_{\mathbb{O}_{\mathrm{RNG}},R_g}(\mathcal{S}) = \lfloor D^{\varepsilon}_{H}(\mathcal{S}\,||\,\mathrm{aff}(\mathbb{F}))\rfloor_{\mathbb{T}}}$$

(The similar relation also holds for resource theory of channels)

- The performance of the black box resource distillation in the general resource theory is also characterized by the composite hypothesis testing divergence

Free operation of q-thermo

Necessary condition for free operations $\mathbb{O}$: $\Lambda(\tau) = \tau, \quad \forall \Lambda \in \mathbb{O}$

(Gibbs-preserving condition)

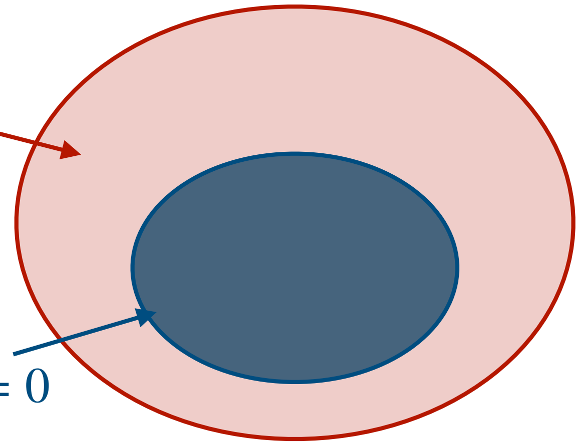
Axiomatic

Gibbs-preserving operation

$$\Lambda(\tau_A) = \tau_B$$

Thermal operation

$$\Lambda_{\mathrm{TO}}(\rho_A) = \mathrm{Tr}_{A+E-B}[U_{AE}(\rho_A \otimes \tau_E)U_{AE}^{\dagger}], \quad [U_{AE}, H_A + H_E] = 0$$



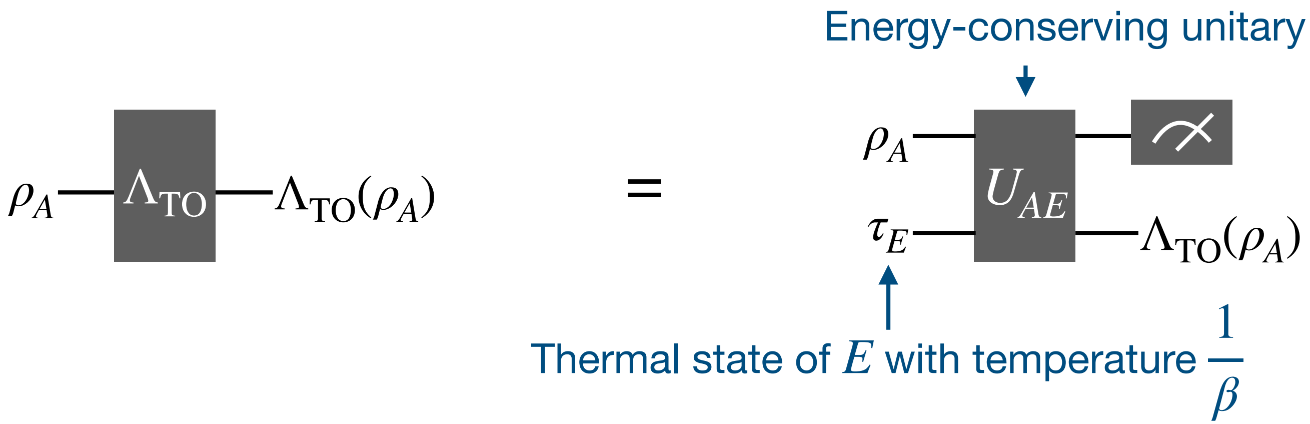
Physically realizable

Free operation of q-thermo

[Horodecki, Oppenheim, Nat. Commun.(2019)]

Thermal Operation: Implementable class of operations

$$\Lambda_{\mathrm{TO}}(\rho_A) = \mathrm{Tr}_{A+E-B}[U_{AE}(\rho_A \otimes \tau_E)U_{AE}^{\dagger}], \quad [U_{AE}, H_A + H_E] = 0.$$



- Gibbs-preserving $\Lambda(\tau_A) = \tau_B$
- Covariant under time-translation $e^{-iH_Bt}\Lambda_{\mathrm{TO}}(\rho_A)e^{iH_Bt} = \Lambda_{\mathrm{TO}}(e^{-iH_At}\rho_A e^{iH_At})$

⇒Cannot create a superposition between different energies from scratch

E.g.) $|E_1\rangle \nrightarrow \dfrac{|E_0\rangle + |E_2\rangle}{\sqrt{2}}$

Free operation of q-thermo

Necessary condition for free operations $\mathbb{O}$: $\Lambda(\tau) = \tau, \quad \forall \Lambda \in \mathbb{O}$

(Gibbs-preserving condition)

Axiomatic

Gibbs-preserving operation

$$\Lambda(\tau_A) = \tau_B$$

$\supsetneq$ [Faist et al., NJP (2015)]

Gibbs-preserving covariant operation

$$\Lambda(\tau_A) = \tau_B, \quad e^{-iH_Bt}\Lambda_{\mathrm{TO}}(\rho_A)e^{iH_Bt} = \Lambda_{\mathrm{TO}}(e^{-iH_At}\rho_A e^{iH_At})$$

$\supsetneq$ [Ćwikliński et al.,PRL (2015)] [Ding et al., PRA(2021)]

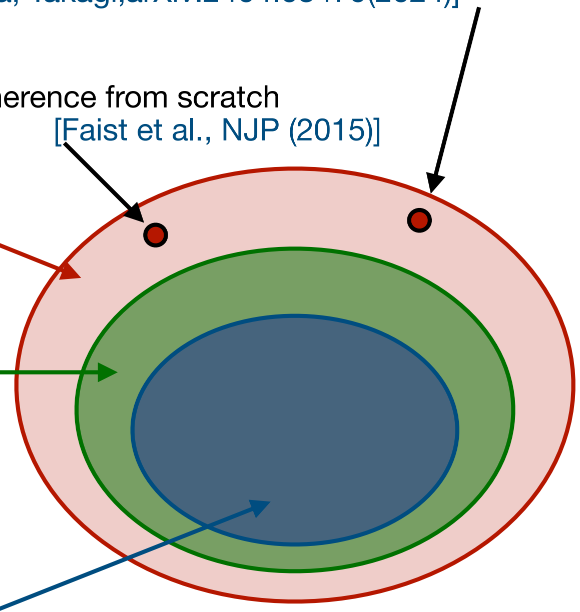
Thermal operation

$$\Lambda_{\mathrm{TO}}(\rho_A) = \mathrm{Tr}_{A+E-B}[U_{AE}(\rho_A \otimes \tau_E)U_{AE}^\dagger], \quad [U_{AE}, H_A + H_E] = 0$$

Physically realizable

Requires an infinite amount of coherence
[Tajima, Takagi,arXiv:2404.03479(2024)]

Creates coherence from scratch
[Faist et al., NJP (2015)]



Free operation of q-thermo

Necessary condition for free operations $\mathbb{O}$: $\Lambda(\tau) = \tau, \quad \forall \Lambda \in \mathbb{O}$
(Gibbs-preserving condition)

Axiomatic

Gibbs-preserving operation

$$\Lambda(\tau_A) = \tau_B$$

$\rotatebox[origin=c]{90}{$\subsetneq$}$ [Faist et al., NJP (2015)]

Gibbs-preserving covariant operation

$$\Lambda(\tau_A) = \tau_B, \quad e^{-iH_B t}\Lambda_{\mathrm{TO}}(\rho_A)e^{iH_B t} = \Lambda_{\mathrm{TO}}(e^{-iH_A t}\rho_A e^{iH_A t})$$

$\rotatebox[origin=c]{90}{$\subsetneq$}$ [Ćwikliński et al.,PRL (2015)] [Ding et al., PRA(2021)]

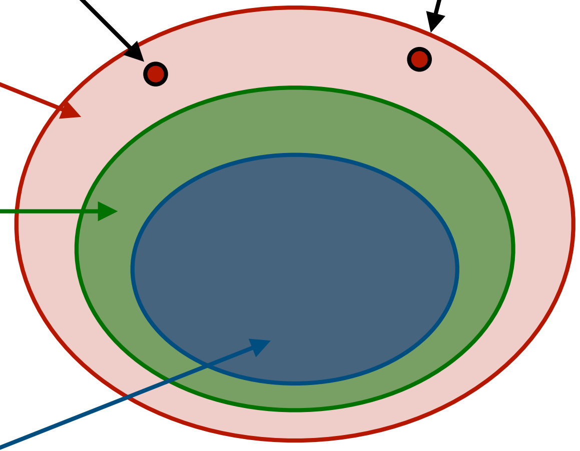
Thermal operation

$$\Lambda_{\mathrm{TO}}(\rho_A) = \mathrm{Tr}_{A+E-B}[U_{AE}(\rho_A \otimes \tau_E)U_{AE}^\dagger], \quad [U_{AE}, H_A + H_E] = 0$$

Physically realizable

Requires an infinite amount of coherence [Tajima, Takagi,arXiv:2404.03479(2024)]

Creates coherence from scratch [Faist et al., NJP (2015)]



Work extraction with Gibbs-preserving covariant operations

Gibbs-preserving covariant operations

→CPTP maps satisfying

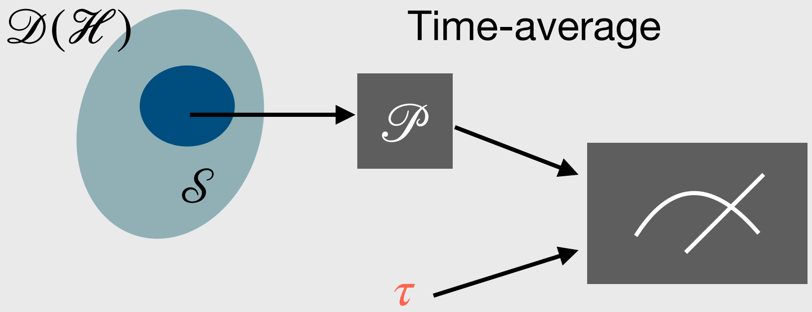
- Gibbs-preserving $\Lambda(\tau_A) = \tau_B$
- Covariant under time-translation $e^{-iH_Bt}\Lambda_{\mathrm{TO}}(\rho_A)e^{iH_Bt} = \Lambda_{\mathrm{TO}}(e^{-iH_At}\rho_A e^{iH_At})$

- Axiomatic but closer to thermal operations

Result

$$\boxed{W^{\varepsilon}_{\mathrm{GPC}}(\mathscr{S}) = k_B T D^{\varepsilon}_{H}(\mathscr{P}(\mathscr{S})\|\tau)}$$

Time-translation covariance corresponds to
the inaccessibility to the information of the coherence



Pinching (decohering) channel: time-average [Tomamichel, Springer(2016)]

$$\mathscr{P}(\rho) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt\ e^{-iHt}\rho e^{iHt} = \sum_{E_i} \Pi_{E_i}\rho\Pi_{E_i}$$

New quantum Stein's lemma

$$W_{\mathrm{GPC}}^{\mathrm{asymp}}(\{\mathcal{S}_n^{\mathrm{TP}}\}) = k_B T \lim_{\varepsilon\to+0} \limsup_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\mathcal{P}(\mathcal{S}^{\mathrm{TP}}{}_n)\|\tau^{\otimes n})$$

The pinching channel breaks the tensor-product structure

→Cannot obtain the limit from the previous results

<u>Result</u>

$$\boxed{\lim_{\varepsilon\to+0} \lim_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\mathcal{P}(\mathcal{S}_n^{\mathrm{TP}})\|\tau^{\otimes n}) = k_B T \min_{\rho\in\mathscr{C}(S)} D(\rho\|\tau)}$$

From this, we have

$$\boxed{W_{\mathrm{GPC}}^{\mathrm{asymp}}(\{\mathcal{S}_n^{\mathrm{TP}}\}_{n=1}^{\infty}) = k_B T \min_{\rho\in\mathscr{C}(S)} D(\rho\|\tau)}$$

$$W_{\mathrm{GPO}}^{\mathrm{asymp}}(\{\mathcal{S}_n^{\mathrm{TP}}\}_n) = k_B T \min_{\rho\in\mathscr{C}(S)} D(\rho\|\tau)$$

Proof idea: Use the technique of adversarial hypothesis testing [Brandão et al., IEEE (2020)]

- A new quantum Stein's lemma motivated by the physical situation
- Gibbs-preserving covariant operations works as well as GPO asymptotically

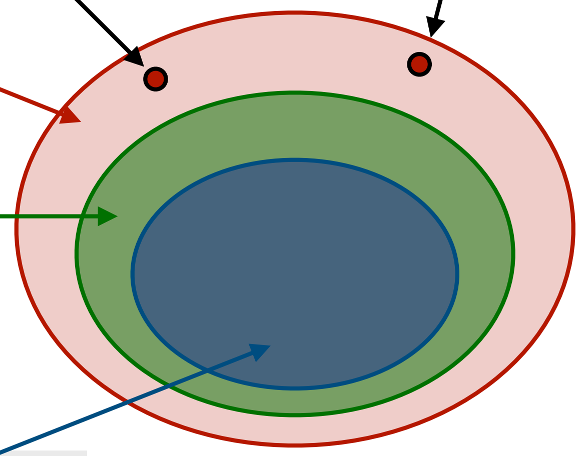
Free operation of q-thermo

Necessary condition for free operations $\mathbb{O}$: $\Lambda(\tau) = \tau, \quad \forall \Lambda \in \mathbb{O}$

(Gibbs-preserving condition)

Requires an infinite amount of coherence
[Tajima, Takagi,arXiv:2404.03479(2024)]

Creates coherence from scratch
[Faist et al., NJP (2015)]



Axiomatic

Gibbs-preserving operation

$$\Lambda(\tau_A) = \tau_B$$

$\subsetneq$ [Faist et al., NJP (2015)]

Gibbs-preserving covariant operation

$$\Lambda(\tau_A) = \tau_B, \; e^{-iH_Bt}\Lambda_{\mathrm{TO}}(\rho_A)e^{iH_Bt} = \Lambda_{\mathrm{TO}}(e^{-iH_At}\rho_A e^{iH_At})$$

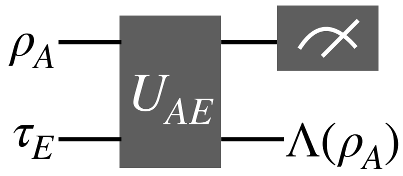
$\subsetneq$ [Ćwikliński et al.,PRL (2015)] [Ding et al., PRA(2021)]

Thermal operation

$$\Lambda_{\mathrm{TO}}(\rho_A) = \mathrm{Tr}_{A+E-B}[U_{AE}(\rho_A \otimes \tau_E)U_{AE}^\dagger], \quad [U_{AE}, H_A + H_E] = 0$$

Physically realizable

Black box work extraction by thermal operation



<u>**Thermal operation**</u>=Realizable class of operation

$$\Lambda_{\mathrm{TO}}(\rho_A) = \mathrm{Tr}_{A+E-B}[U_{AE}(\rho_A \otimes \tau_E)U_{AE}^{\dagger}], \quad [U_{AE}, H_A + H_E] = 0$$

- State-aware work extraction protocol by thermal operation

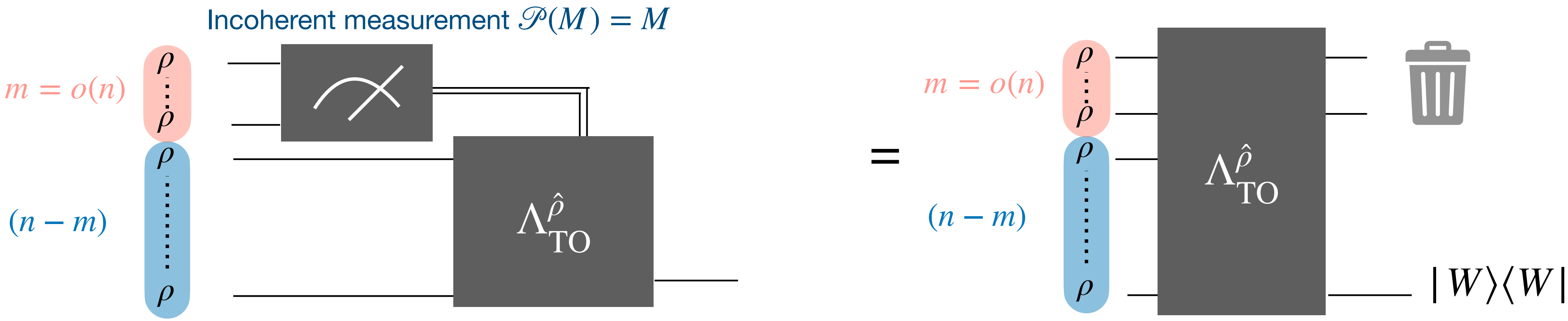
$$\boxed{W_{\mathrm{TO}}^{\mathrm{asymp}}(\rho) = D(\rho\,||\,\tau)}$$

[Brandão et al., PRL(2013)]

<u>Result</u>

$$\boxed{\mathcal{S}_n = \{\rho^{\otimes n} \mid \rho \in S,\ |S| < \infty\} \Rightarrow W_{\mathrm{TO}}^{\mathrm{asymp}}(\{\mathcal{S}_n\}_{n=1}^{\infty}) = \min_{\rho \in S} D(\rho\,||\,\tau)}$$

<u>Proof idea</u>: Perform the state tomography on the sublinear amount of states



Conclusion

- Introduced the framework of the black box work extraction
- Gave an operational interpretation of the composite hypothesis testing divergence with composite hypothesis in 1st argument (Also in Kun Fang's talk)
- Showed a new quantum Stein's lemma with a composite hypothesis motivated by physical setting

Outlook

- Complete characterization of the black box work extraction with thermal operations
- Can other quantum information theoretic tasks be done state-agnostically?
  - State-agnostic resource distillation in other QRT?
- Other physically motivated quantum Stein's lemma?

Thank you!