Temporal correlation in quantum states as a resource

Seok Hyung Lie UNIST

Quantum Resources 2025

1. **What is a quantum state over time (QSOT)?**

Symmetry of space and time in probabilistic theories?

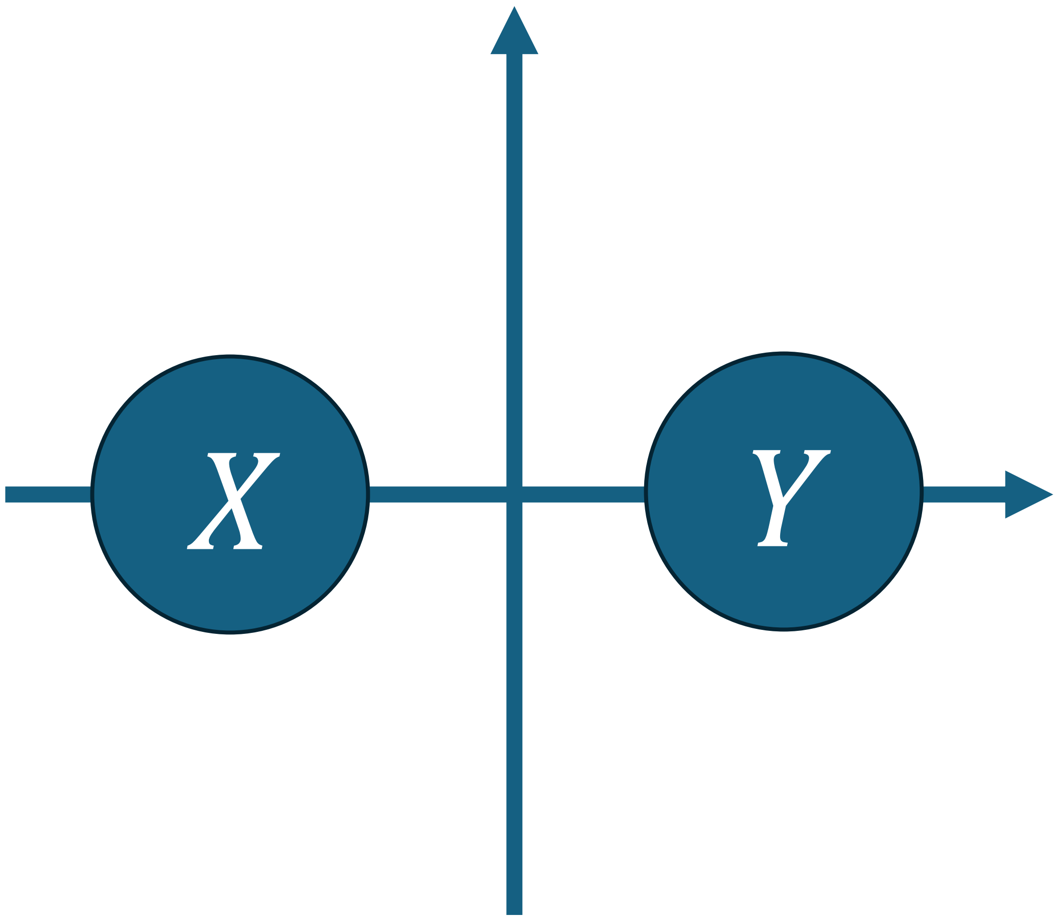
In classical theories…

Positions of events X and Y in spacetime?

$$P(X, Y)$$

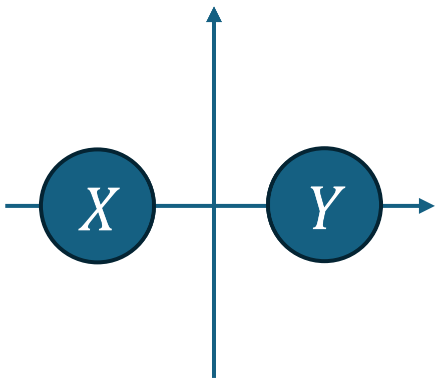
Space-like?

Time-like?



Their correlation is still described with $P(X, Y)$

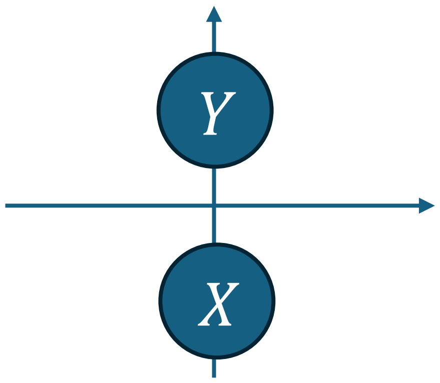
What about in quantum theory?



Space-like

$$\rho_{XY}$$

(Multipartite)
quantum states



Time-like

$$\sigma_Y = \Phi(\rho_X)$$

Quantum channels

Is sensitivity to spatiotemporal structure
an inherent property of quantum theory?

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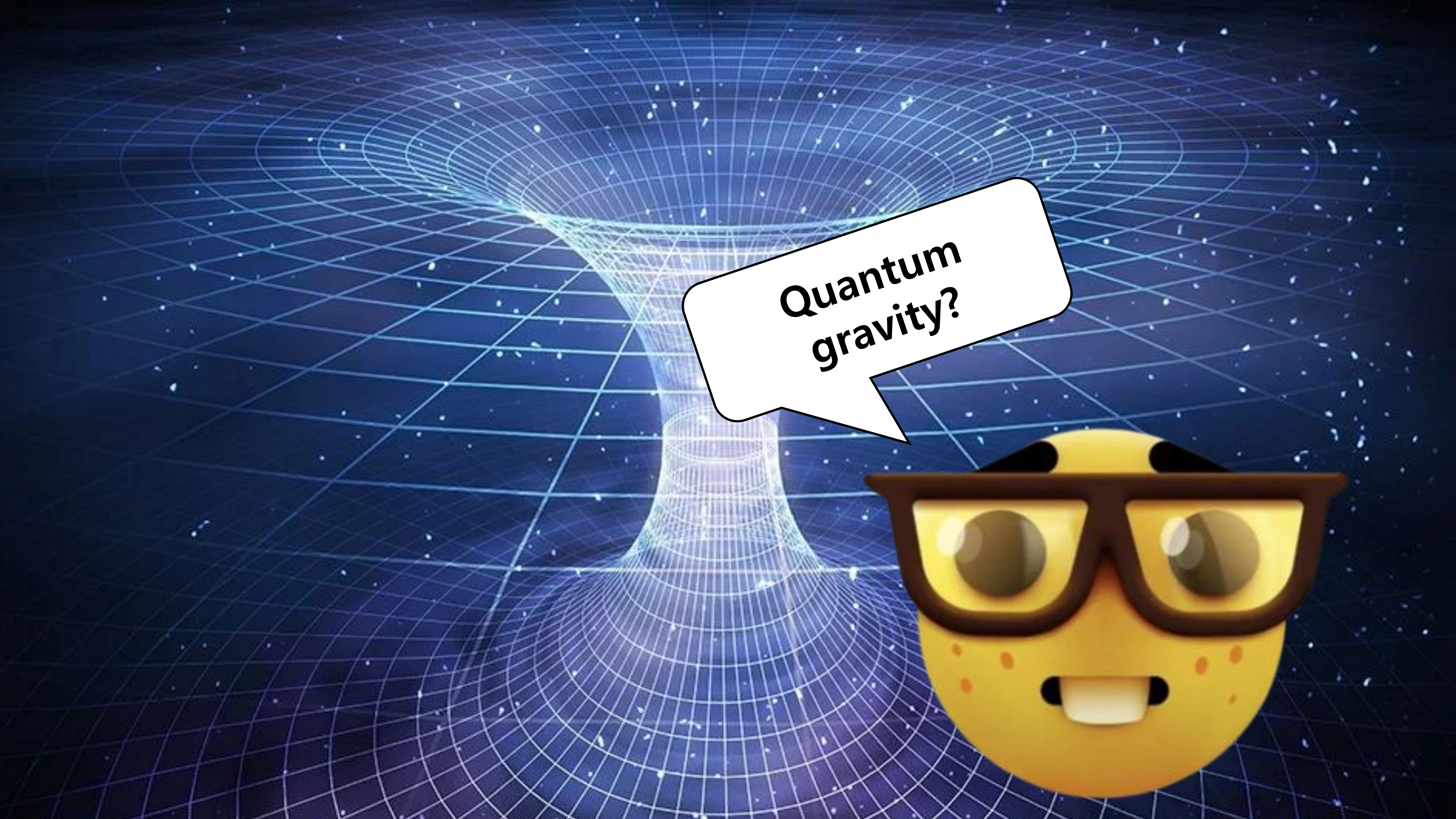
Is sensitivity to spatiotemporal structure an inherent property of quantum theory?


Quantum gravity?

PHYSICAL REVIEW A
covering atomic, molecular, and optical physics and quantum information

Towards a formulation of quantum theory as a causally neutral theory of Bayesian inference

M. S. Leifer and Robert W. Spekkens

Phys. Rev. A **88**, 052130 – Published 27 November 2013

Can we construct quantum state over time?

$$(\rho_A, \Phi_{B|A}) \Rightarrow \boxed{\Phi_{B|A} \star \rho_A}$$

"Quantum state over time function"

SCIENTIFIC REPORTS

OPEN

Quantum correlations which imply causation

Joseph F. Fitzsimons[1,2], Jonathan A. Jones[3] & Vlatko Vedral[2,3,4]

$$\Phi_{B|A} \star \rho_A \overset{?}{\geq} 0$$

$$\Phi_{B|A} \star \rho_A \ngeq 0$$

PROCEEDINGS OF THE ROYAL SOCIETY A

MATHEMATICAL, PHYSICAL AND ENGINEERING SCIENCES

Can a quantum state over time resemble a quantum state at a single time?

Dominic Horsman, Chris Heunen, Matthew F. Pusey, Jonathan Barrett and Robert W. Spekkens

Published: 20 September 2017 | https://doi.org/10.1098/rspa.2017.0395

(a) Hermiticity
(b) Preservation of probabilistic mixtures
(c) Preservation of classical limit
(d) Preservation of marginal states
(e) Compositionality

(a) He
(b) Pr … ixtures
(c) Pre
(d) Pr … s
(e) Co

It turned out that the criteria were translated into too strong mathematical conditions

PROCEEDINGS OF THE ROYAL SOCIETY A

MATHEMATICAL, PHYSICAL AND ENGINEERING SCIENCES

On quantum states over time

James Fullwood and Arthur J. Parzygnat

Published: 03 August 2022 | https://doi.org/10.1098/rspa.2022.0104

$\{A, B\} := \frac{1}{2}(AB + BA)$

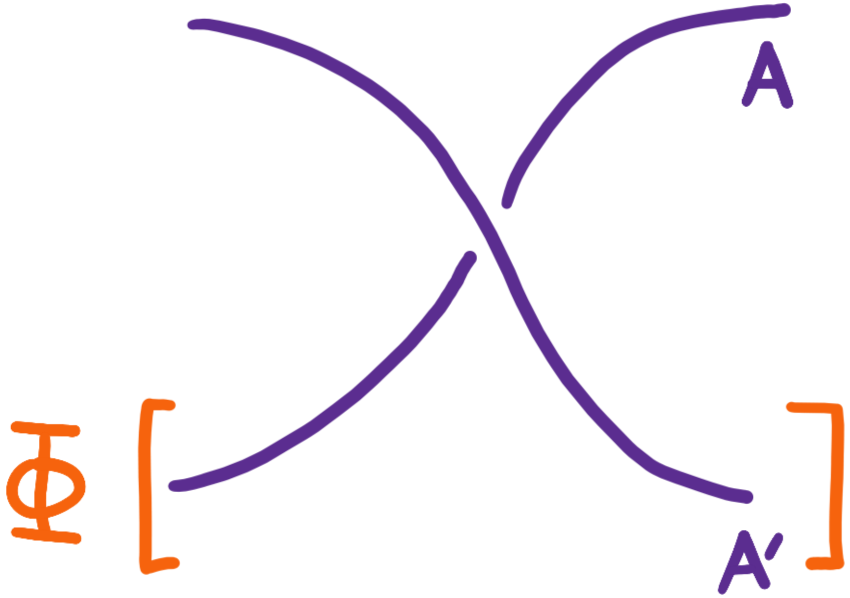
$$\Phi_{B|A} \star_{FP} \rho_A = \frac{1}{2}\{\rho_A \otimes I_B, D[\Phi_{B|A}]\}$$

where $D[\Phi_{B|A}] = \mathrm{id}_A \otimes \Phi_{\mathrm{B|A'}}(\mathrm{F}_{\mathrm{AA'}})$

Jamiołkowski isomorphism

$$\Phi_{B|A} \star_{FP} \rho_A = \frac{1}{2}\{\rho_A \otimes I_B, D[\Phi_{B|A}]\}$$

where $D[\Phi_{B|A}] = \mathrm{id}_A \otimes \boxed{\Phi_{\mathrm{B|A'}}}\boxed{(\mathrm{F}_{\mathrm{AA'}})}$



$$\Phi_{B|A} \star_{FP} \rho_A = \frac{1}{2}\{\boxed{\rho_A \otimes I_B,} D[\Phi_{B|A}]\}$$

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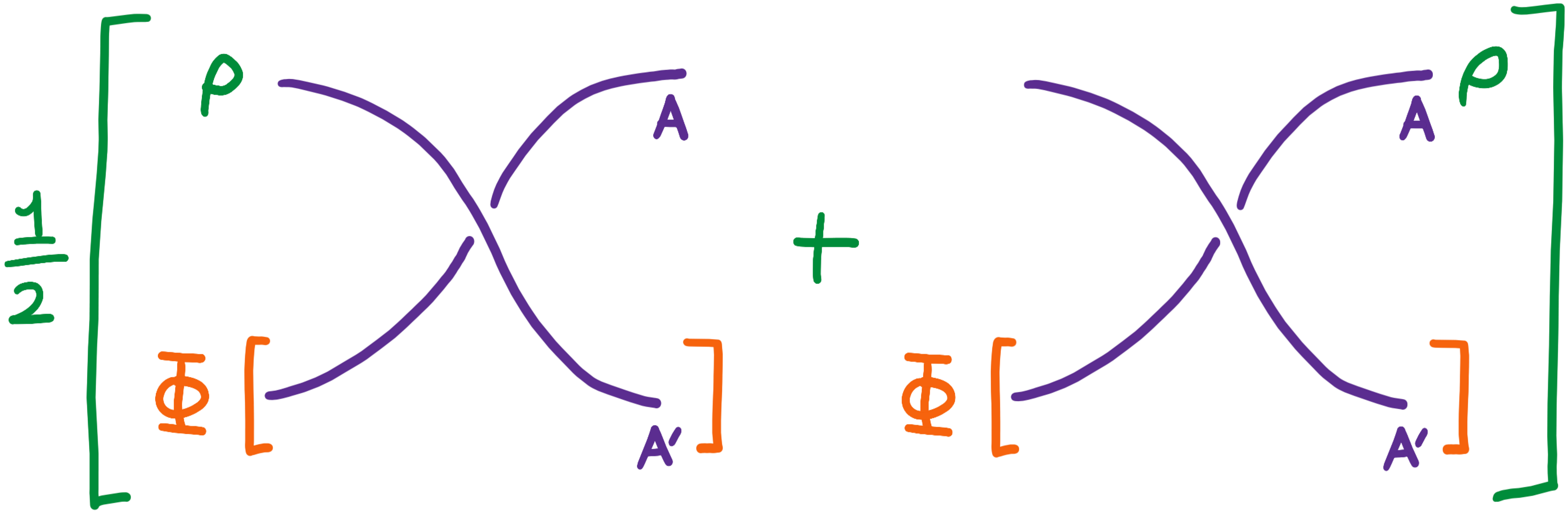

$$\Phi_{B|A} \star_{FP} \rho_A = \frac{1}{2}\{\rho_A \otimes I_B, D[\Phi_{B|A}]\}$$

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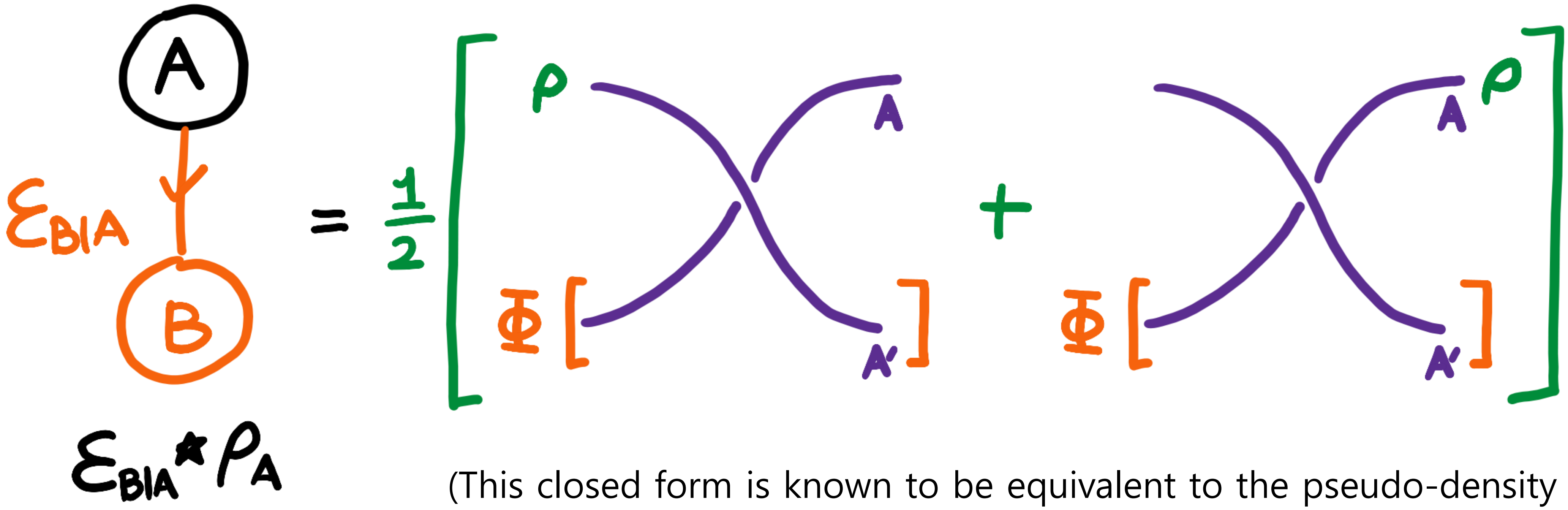

(This closed form is known to be equivalent to the pseudo-density operator (PDO) when limited to multi-qubit systems)

PRX QUANTUM
a Physical Review journal

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From Time-Reversal Symmetry to Quantum Bayes' Rules

Arthur J. Parzygnat and James Fullwood
PRX Quantum **4**, 020334 – Published 2 June 2023

TABLE II. The many state-over-time functions appearing in this work, along with their formulas, properties satisfied, and associated Bayes maps. The axioms are Hermiticity (P1), block-positivity (P2), positivity (P3), state linearity (P4), process linearity (P5), the classical limit (P7), and associativity A (bilinearity has been removed from the table to avoid redundancy). The $*$ for Ohya's compound state over time is because the classical limit is satisfied for density matrices with no repeating eigenvalues. Note that we do not fully define Ohya's compound state over time for arbitrary CPTP maps between multimatrix algebras (this will be addressed in future work, along with additional examples of state-over-time functions). The question mark represents the fact that we have not yet determined whether the given axiom is satisfied.

Name (page ref.)	State over time $\mathcal{E}\star\rho$	P1	P2	P3	P4	P5	P7	A	Bayes map $\mathcal{E}^{\star}_{\rho}$
Uncorrelated (7)	$\rho\otimes\mathcal{E}(\rho)$	✓	✓	✓	✗	✓	✗	✗	Any CPTP such that $\mathcal{E}^{\star}_{\rho}(\mathcal{E}(\rho))=\rho$
Ohya compound (7)	$\sum_{\alpha}\lambda_{\alpha}P_{\alpha}\otimes\mathcal{E}\left(\frac{P_{\alpha}}{\mathrm{tr}(P_{\alpha})}\right)$	✓	✓	✓	✗	✓	$*$	?	Not computed here
Leifer-Spekkens (7)	$(\sqrt{\rho}\otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}](\sqrt{\rho}\otimes 1_{\mathcal{B}})$	✓	✓	✗	✗	✓	✓	✗	Petz map $\mathscr{R}_{\rho,\mathcal{E}}:=\mathrm{Ad}_{\rho^{1/2}}\circ\mathcal{E}^{*}\circ\mathrm{Ad}_{\mathcal{E}(\rho)^{-1/2}}$
t-rotated (8)	$(\rho^{1/2-it}\otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}](\rho^{1/2+it}\otimes 1_{\mathcal{B}})$	✓	✓	✗	✗	✓	✓	✗	Rotated Petz map $\mathrm{Ad}_{\rho^{-it}}\circ\mathscr{R}_{\rho,\mathcal{E}}\circ\mathrm{Ad}_{\mathcal{E}(\rho)^{it}}$
STH (8)	$(U_{\rho}^{\dagger}\rho^{1/2}\otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}](\rho^{1/2}U_{\rho}\otimes 1_{\mathcal{B}})$	✓	✓	✗	✗	✓	✓	✗	$\mathrm{Ad}_{U_{\rho}^{\dagger}}\circ\mathscr{R}_{\rho,\mathcal{E}}\circ\mathrm{Ad}_{U_{\mathcal{E}(\rho)}}$
Symmetric bloom (11)	$\frac{1}{2}\{\rho\otimes 1_{\mathcal{B}},\mathscr{D}[\mathcal{E}]\}$	✓	✗	✗	✓	✓	✓	✓	$\lvert w_k\rangle\langle w_l\rvert\mapsto(q_k+q_l)^{-1}\{\rho,\mathcal{E}^{*}(\lvert w_k\rangle\langle w_l\rvert)\}$
Right bloom (13)	$(\rho\otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}]$ (e.g., two-state)	✗	✗	✗	✓	✓	✓	✓	$B\mapsto\rho\mathcal{E}^{*}(\mathcal{E}(\rho)^{-1}B)$ (e.g., weak values)
Left bloom (15)	$\mathscr{D}[\mathcal{E}](\rho\otimes 1_{\mathcal{B}})$ (e.g., correlator)	✗	✗	✗	✓	✓	✓	✓	$B\mapsto\mathcal{E}^{*}(B\mathcal{E}(\rho)^{-1})\rho$

PHYSICAL REVIEW RESEARCH

Open Access

Quantum state over time is unique

Seok Hyung Lie and Nelly H. Y. Ng

Phys. Rev. Research **6**, 033144 – Published 6 August 2024

Quantum state over time (QSOT) function

A function $\star: \mathcal{C}(A,B) \times \mathcal{S}(A) \to A \otimes B$ that maps $(\Phi_{\mathrm{B|A}}, \rho_A)$ to $\Phi_{B|A} \star \rho_A$ is a QSOT function if

$$\mathrm{Tr}_B \Phi_{B|A} \star \rho_A = \rho_A$$
$$\mathrm{Tr}_A \Phi_{B|A} \star \rho_A = \Phi(\rho)_B$$

Axiom (E): Completeness
For any quantum state over spacetime ρ_{AE} with two arbitrary regions A and E in spacetime, and any quantum channel $\mathcal{E}_{B|A}$, the action of QSOT function on a subsystem $\mathcal{E}_{B|A} \star \rho_{AE}$ can be defined and has the following properties: For any completely positive trace non-increasing opera

$\mathcal{I}_E[\mathcal{E}_{B|A} \star \rho$

Axiom (T): Time reversal symmetry
A state over time corresponding to the trivial evolution should be symmetric under the time reversal transformation, i.e. $F_{AB}(\mathrm{id}_{B|A} \star \rho_A)F_{AB} = \mathrm{id}_{B|A} \star \rho_A$, for

notes the swap gate be-

Axiom (H) (Hermiticity) For any quantum channel and state $\mathcal{E}_{B|A}, \rho_A$, the state over time $\mathcal{E}_{B|A} \star \rho_A$ must be Hermitian.

Axiom (P): Composi
A QSOT function should be compatible with composition of quantum channels. In other words, for any two quantum channels $\mathcal{E}_{B|A}$ and $\mathcal{F}_{C|B}$ we have

$$\mathrm{Tr}_B\left[\mathcal{F}_{C|B} \star (\mathcal{E}_{B|A} \star \rho_A)\right] = (\mathcal{F} \circ \mathcal{E})_{C|A} \star \rho_A. \quad (5)$$

ditionability (informal)
When the input state and the channel are prepared in an ensemble $\{\lambda_i, \pi_{A_i}, \mathcal{E}_{B|A_i}\}$, then the corresponding QSOT is given as $\mathcal{E}_{B|A} \star (\sum_i \lambda_i \pi_{A_i}) = \sum_i \lambda_i \mathcal{E}_{B|A_i} \star \pi_{A_i}$, where $\mathcal{E}_{B|A} = \sum_i \mathcal{E}_{B|A_i}$.

$$\mathcal{E}_{B|A} \star_{FP} \rho_A = \frac{1}{2}\{\rho_A \otimes I_B, D[\mathcal{E}_{B|A}]\}$$

Axiom (QC): Quantum conditionability

For every state $\rho \in \mathfrak{S}(A)$, there exists a state-rendering function Θ_ρ [17,40,41] on $\mathfrak{B}(A)$ such that

$$\mathcal{E}_{B|A} \star \rho_A = (\Theta_\rho \otimes \mathrm{id}_B)(\mathcal{E}_{B|A} \star \mathbb{1}_A) \tag{13}$$

for all $\mathcal{E} \in \mathfrak{C}(A, B)$, where Θ_ρ is linear, and for any $M \in \mathfrak{B}(A)$, whenever $[\rho, M] = 0$, we have $\Theta_\rho(M) = \rho M$.

(This part requires reduction to classical state rendering function, which may warrant a separate axiom.)

$$P(X,Y) = P(Y|X)P(X)$$

"state-rendering"

$$\rho_{AB} = \rho_{B|A} \star \rho_A \;???$$

$$\mathcal{E}_{B|A} \star \rho_A = (\Theta_\rho \otimes \mathbb{1}_B)[\mathcal{E}_{B|A} \star \mathbb{1}_A]$$

"state-rendering function"

"conditional quantum state"

Quantum Physics

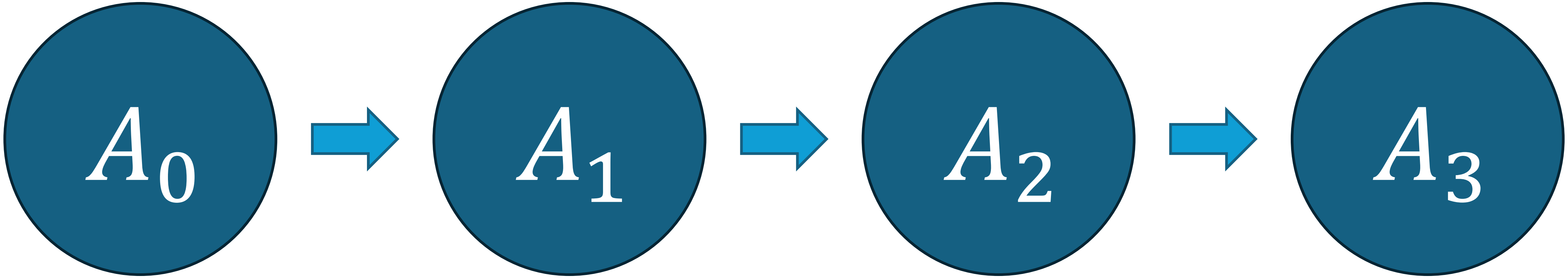
[Submitted on 30 Oct 2024]

Unique multipartite extension of quantum states over time

Seok Hyung Lie, James Fullwood

The quantum state over time formalism provides an extension of the density operator formalism into the time domain, so that quantum correlations across both space and time may be treated with a common mathematical formalism. While bipartite quantum states over time have been uniquely characterized from various perspectives, it is not immediately clear how to extend the uniqueness result to multipartite temporal scenarios, such as those considered in the context of Legget-Garg inequalities. In this Letter, we show that two simple assumptions uniquely single out a multipartite extension of bipartite quantum states over time, namely, linearity in the initial state and a quantum analog of conditionability for multipartite probability distributions. As a direct consequence of our uniqueness result we arrive at a canonical multipartite extension of Kirkwood-Dirac type quasi-probability distributions, and we conclude by showing how our result yields a new characterization of quantum Markovianity.

N-chains



$$\boldsymbol{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n) \in \mathbf{CPTP}(A_0, A_1, \dots, A_n)$$
$$\Leftrightarrow \mathcal{E}_i \in \mathbf{CPTP}(A_0, A_1, \dots, A_n) \text{ for all } i = 1, \dots, n$$

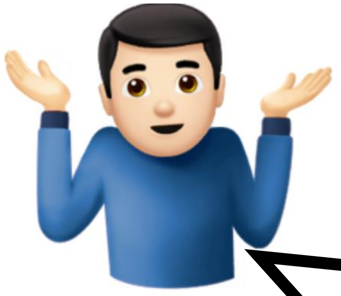
QSOT function, QSOT product,
spatiotemporal product, start product…
all same

Definition 1 (Quantum state over time)**.** A *spatiotemporal product* (or $\star$*-product*) is a binary operation that maps every pair $(\boldsymbol{\mathcal{E}}, \rho) \in \mathbf{CPTP}(A_0, \ldots, A_n) \times \mathfrak{S}(A_0)$ to an $(n+1)$-partite operator $\boldsymbol{\mathcal{E}} \star \rho$ on $A_0 \cdots A_n$ such that

$$\mathrm{Tr}_{A_0}[\boldsymbol{\mathcal{E}} \star \rho] = \underline{\boldsymbol{\mathcal{E}}} \star \mathcal{E}_1(\rho) \quad \text{and} \quad \mathrm{Tr}_{A_n}[\boldsymbol{\mathcal{E}} \star \rho] = \overline{\boldsymbol{\mathcal{E}}} \star \rho\,.$$

Truncations of n-chains

$$\underline{\boldsymbol{\mathcal{E}}} := (\mathcal{E}_2, \mathcal{E}_3, \ldots, \mathcal{E}_n) \quad \text{and} \quad \overline{\boldsymbol{\mathcal{E}}} := (\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{n-1})\,.$$



Common question:

We have a uniqueness result for bipartite QSOTs;
Why can't we just extend it to the multipartite setting?

$$\mathcal{E} \star \rho = \mathcal{E}_n \star (\mathcal{E}_{n-1} \star (\cdots \star (\mathcal{E}_1 \star \rho)))$$

Well, yeah, you **CAN** do that.
But the problem is, this is by no means the **unique** multipartite extension of $\star$.

There are **exponentially many** n-chain products that reduces to the FP product.

$$\mathcal{E} \star_{\mathbf{C}} \rho = \mathcal{E}_n \star_{C_n} (\mathcal{E}_{n-1} \star_{C_{n-1}} (\cdots (\mathcal{E}_1 \star_{C_1} \rho)))$$

$$\mathcal{E} \star \rho = \frac{1}{2}(\mathcal{E} \star_{\mathbf{C}} \rho + \mathcal{E} \star_{\tilde{\mathbf{C}}} \rho)$$

Where $\boldsymbol{C} = (C_1, C_2, \ldots, C_n) \in \{L, R\}^n$ and $\widetilde{\boldsymbol{C}}$ is the opposite of $\boldsymbol{C}$.

☞ **Uniqueness problem for multipartite QSOT is highly nontrivial!**

Which axioms should we assume?

As natural as possible, as few as possible.

1. State-linearity

We want our QSOT function $\mathcal{E} \star \rho$ to be linear in ρ.

Good : consistency with statistical reasoning

Easily extends to the multipartite setting

2. Conditionability

We want our QSOT function $\mathcal{E} \star \rho$ to behave similarly with classical probability distributions.

Especially: We want to be able to '**condition**' on an initial state

$$P(x_0, \ldots, x_n) = P(x_0) \cdot P(x_1, \ldots, x_n | x_0)$$

For QSOT, it amounts to the following assumption.

Definition 2 (Conditionability)**.** A $\star$-product is said to be *conditionable* if and only if for every state $\rho \in \mathfrak{S}(A_0)$, there exists a linear map $\Theta_\rho : A_0 \to A_0$ such that for every n-chain $\boldsymbol{\mathcal{E}} \in \mathbf{CPTP}(A_0, \ldots, A_n)$,

$$\boldsymbol{\mathcal{E}} \star \rho = (\Theta_\rho \otimes \mathrm{id}_{A_1 \cdots A_n})(\boldsymbol{\mathcal{E}} \star \mathbb{1}_{A_0})\,. \tag{6}$$

Surprisingly, these two assumptions are enough
to prove the uniqueness of multipartite extension.

Theorem 1 (Unique multi-partite extension of QSOTs). If a $\star$-product is conditionable and convex-linear in ρ, then it satisfies the iterative formula (3) for every $(\boldsymbol{\mathcal{E}}, \rho) \in \mathbf{CPTP}(A_0, \ldots, A_n) \times \mathfrak{S}(A_0)$.

$$\boldsymbol{\mathcal{E}} \star \rho = \mathcal{E}_n \star (\mathcal{E}_{n-1} \star (\cdots \star (\mathcal{E}_1 \star \rho))) \qquad (3)$$

The Markovian extension of quantum state over time!

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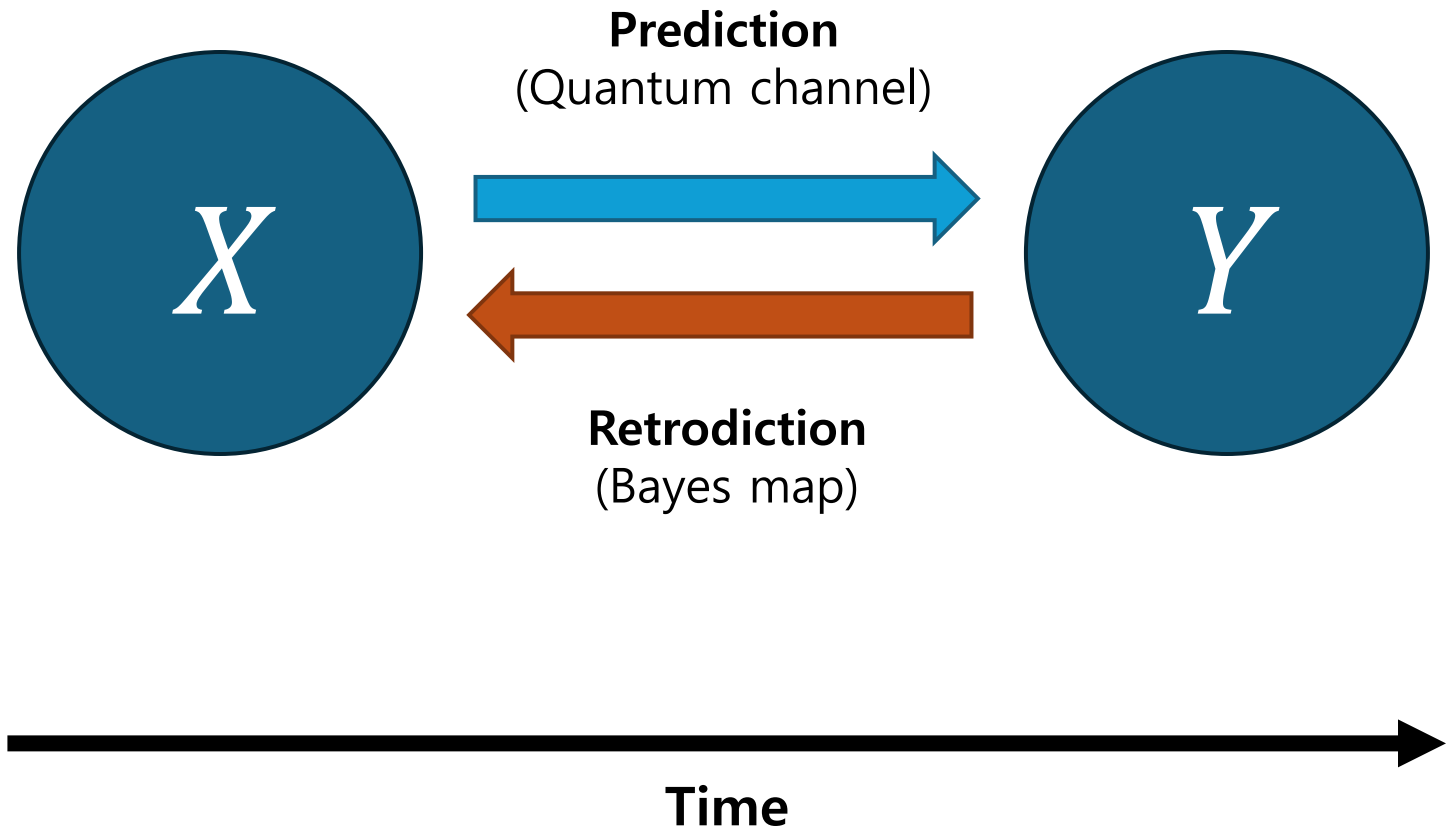
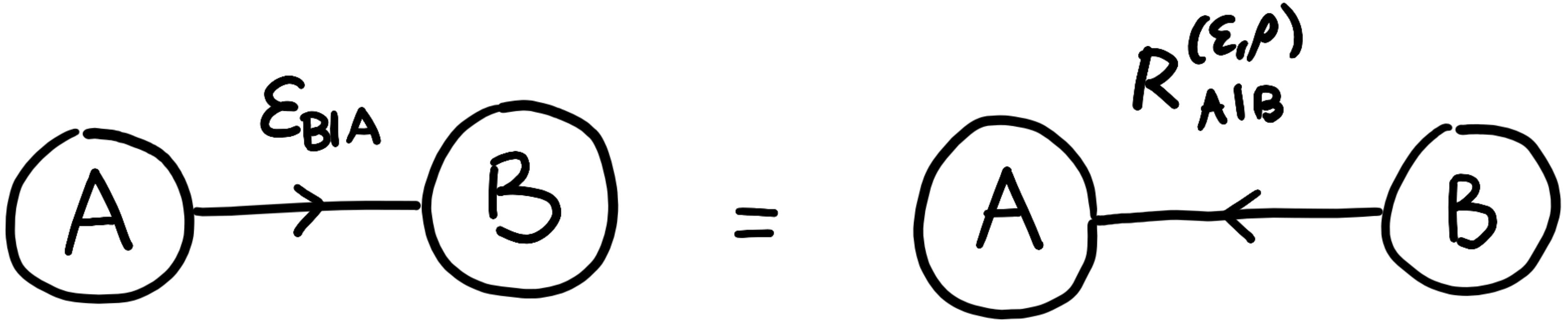
From Time-Reversal Symmetry to Quantum Bayes' Rules

Arthur J. Parzygnat and James Fullwood

PRX Quantum **4**, 020334 – Published 2 June 2023


Prediction
(Quantum channel)
X
Y
Retrodiction
(Bayes map)
Time



$$\mathcal{E}_{B|A} \star \rho_A = R_{A|B}^{(\mathcal{E},\rho)} \star \mathcal{E}(\rho_A)$$

$$R_{A|B}^{(\mathcal{E},\rho)} \coloneqq \Theta_\rho \circ \mathcal{E}^\dagger \circ \Theta_\rho^{-1}$$

TABLE II. The many state-over-time functions appearing in this work, along with their formulas, properties satisfied, and associated Bayes maps. The axioms are Hermiticity (P1), block-positivity (P2), positivity (P3), state linearity (P4), process linearity (P5), the classical limit (P7), and associativity A (bilinearity has been removed from the table to avoid redundancy). The $*$ for Ohya's compound state over time is because the classical limit is satisfied for density matrices with no repeating eigenvalues. Note that we do not fully define Ohya's compound state over time for arbitrary CPTP maps between multimatrix algebras (this will be addressed in future work, along with additional examples of state-over-time functions). The question mark represents the fact that we have not yet determined whether the given axiom is satisfied.

Name (page ref.)	State over time $\mathcal{E}\star\rho$	P1	P2	P3	P4	P5	P7	A	Bayes map $\mathcal{E}^{\star}_{\rho}$
Uncorrelated (7)	$\rho\otimes\mathcal{E}(\rho)$	✓	✓	✓	✗	✓	✗	✗	Any CPTP such that $\mathcal{E}^{\star}_{\rho}(\mathcal{E}(\rho))=\rho$
Ohya compound (7)	$\sum_{\alpha}\lambda_{\alpha}P_{\alpha}\otimes\mathcal{E}\left(\frac{P_{\alpha}}{\mathrm{tr}(P_{\alpha})}\right)$	✓	✓	✓	✗	✓	$*$	?	Not computed here
Leifer-Spekkens (7)	$(\sqrt{\rho}\otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}](\sqrt{\rho}\otimes 1_{\mathcal{B}})$	✓	✓	✗	✗	✓	✓	✗	Petz map $\mathscr{R}_{\rho,\mathcal{E}}:=\mathrm{Ad}_{\rho^{1/2}}\circ\mathcal{E}^{*}\circ\mathrm{Ad}_{\mathcal{E}(\rho)^{-1/2}}$
t-rotated (8)	$(\rho^{1/2-it}\otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}](\rho^{1/2+it}\otimes 1_{\mathcal{B}})$	✓	✓	✗	✗	✓	✓	✗	Rotated Petz map $\mathrm{Ad}_{\rho^{-it}}\circ\mathscr{R}_{\rho,\mathcal{E}}\circ\mathrm{Ad}_{\mathcal{E}(\rho)^{it}}$
STH (8)	$(U_{\rho}^{\dagger}\rho^{1/2}\otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}](\rho^{1/2}U_{\rho}\otimes 1_{\mathcal{B}})$	✓	✓	✗	✗	✓	✓	✗	$\mathrm{Ad}_{U_{\rho}^{\dagger}}\circ\mathscr{R}_{\rho,\mathcal{E}}\circ\mathrm{Ad}_{U_{\mathcal{E}(\rho)}}$
Symmetric bloom (11)	$\frac{1}{2}\{\rho\otimes 1_{\mathcal{B}},\mathscr{D}[\mathcal{E}]\}$	✓	✗	✗	✓	✓	✓	✓	$\lvert w_k\rangle\langle w_l\rvert\mapsto(q_k+q_l)^{-1}\{\rho,\mathcal{E}^{*}(\lvert w_k\rangle\langle w_l\rvert)\}$
Right bloom (13)	$(\rho\otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}]$ (e.g., two-state)	✗	✗	✗	✓	✓	✓	✓	$B\mapsto\rho\mathcal{E}^{*}(\mathcal{E}(\rho)^{-1}B)$ (e.g., weak values)
Left bloom (15)	$\mathscr{D}[\mathcal{E}](\rho\otimes 1_{\mathcal{B}})$ (e.g., correlator)	✗	✗	✗	✓	✓	✓	✓	$B\mapsto\mathcal{E}^{*}(B\mathcal{E}(\rho)^{-1})\rho$

✗ Any CPTP such that $\mathcal{E}^\star_\rho(\mathcal{E}(\rho)) = \rho$

? Not computed here

✗ $\mathrm{Ad}_{\mathcal{E}(\rho)^{-1/2}}$

✗ $\circ\, \mathrm{Ad}_{\mathcal{E}(\rho)^{it}}$

✗

✓ $|w_k\rangle\langle w_l| \mapsto (q_k + q_l) \ldots \{\rho, \mathcal{E}^*(|w_k\rangle\langle w_l|)\}$

✓ $B \mapsto \rho\mathcal{E}^*\left(\mathcal{E}(\rho)^{-1}B\right)$ (e.g., weak values)

✓ $B \mapsto \mathcal{E}^*\left(B\mathcal{E}(\rho)^{-1}\right)\rho$

1. HPTP map in general

(which is okay because quantum state in the past need not be positive and Bayesian inference has never a physical process)

2. Linear in prior

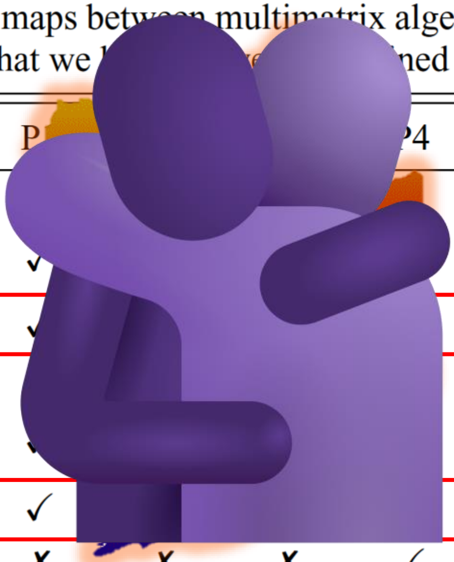
(which is crucial for statistical reasoning)

3. Can update certainty

(which shouldn't be surprising because we already know $S(A) = 0$ but $S(A|B) < 0$ is possible)

(Not necessarily, I would say)

TABLE II. The many state-over-time functions appearing in this work, along with their formulas, properties satisfied, and associated Bayes maps. The axioms are Hermiticity (P1), block-positivity (P2), positivity (P3), state linearity (P4), process linearity (P5), the classical limit (P7), and associativity A (bilinearity has been removed from the table to avoid redundancy). The $*$ for Ohya's compound state over time is because the classical limit is satisfied for density matrices with no repeating eigenvalues. Note that we do not fully define Ohya's compound state over time for arbitrary CPTP maps between multimatrix algebras (this will be addressed in future work, along with additional examples of state-over-time functions). The question mark represents the fact that we [illegible]ned whether the given axiom is satisfied.



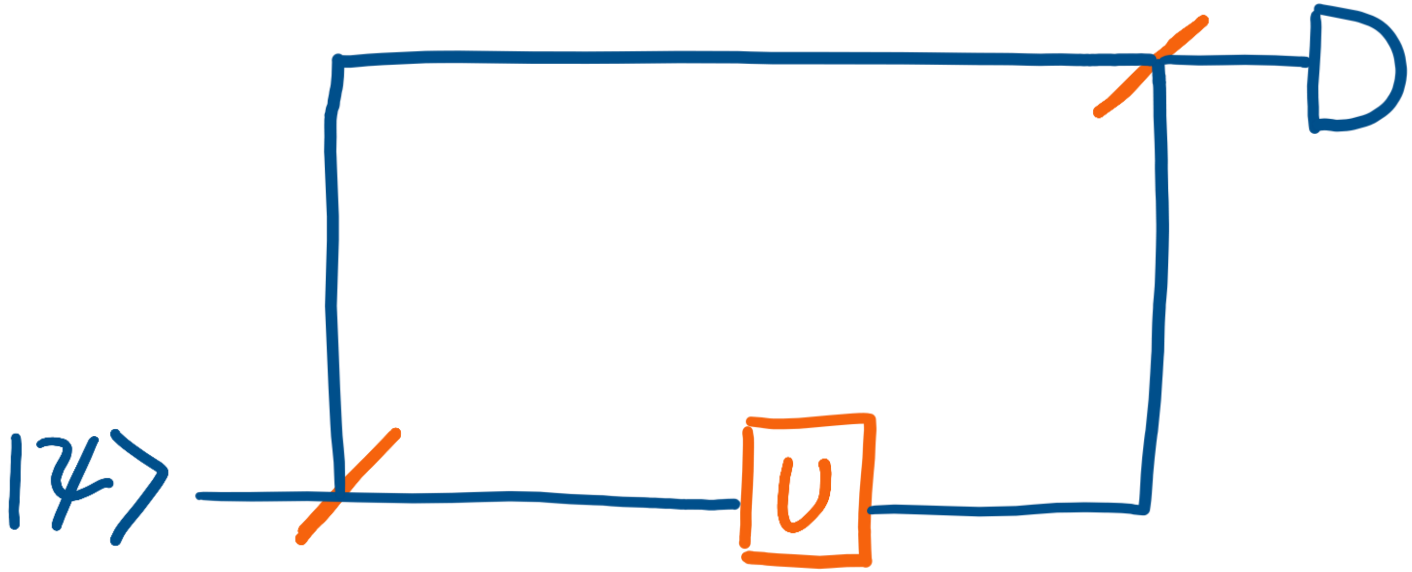
Name (page ref.)	State over time $\mathcal{E} \star \rho$	P[illegible]	[illegible]	[illegible]	[illegible]4	P5	P7	A	Bayes map $\mathcal{E}^{\star}_{\rho}$
Uncorrelated (7)	$\rho \otimes \mathcal{E}(\rho)$	[illegible]	[illegible]	[illegible]	[illegible]	✓	✗	✗	Any CPTP such that $\mathcal{E}^{\star}_{\rho}(\mathcal{E}(\rho)) = \rho$
Ohya compound (7)	$\sum_{\alpha} \lambda_{\alpha} P_{\alpha} \otimes \mathcal{E}\left(\frac{P_{\alpha}}{\mathrm{tr}(P_{\alpha})}\right)$	✓	[illegible]	[illegible]	[illegible]	✓	$*$	?	Not computed here
Leifer-Spekkens (7)	$(\sqrt{\rho} \otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}](\sqrt{\rho} \otimes 1_{\mathcal{B}})$	✓	[illegible]	[illegible]	[illegible]	✓	✓	✗	Petz map $\mathscr{R}_{\rho,\mathcal{E}} := \mathrm{Ad}_{\rho^{1/2}} \circ \mathcal{E}^{*} \circ \mathrm{Ad}_{\mathcal{E}(\rho)^{-1/2}}$
t-rotated (8)	$(\rho^{1/2-it} \otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}](\rho^{1/2+it} \otimes 1_{\mathcal{B}})$	[illegible]	[illegible]	[illegible]	[illegible]	✓	✓	✗	Rotated Petz map $\mathrm{Ad}_{\rho^{-it}} \circ \mathscr{R}_{\rho,\mathcal{E}} \circ \mathrm{Ad}_{\mathcal{E}(\rho)^{it}}$
STH (8)	$(U_{\rho}^{\dagger}\rho^{1/2} \otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}](\rho^{1/2}U_{\rho} \otimes 1_{\mathcal{B}})$	✓	[illegible]	[illegible]	[illegible]	✓	✓	✗	$\mathrm{Ad}_{U_{\rho}^{\dagger}} \circ \mathscr{R}_{\rho,\mathcal{E}} \circ \mathrm{Ad}_{U_{\mathcal{E}(\rho)}}$
Symmetric bloom (11)	$\frac{1}{2}\{\rho \otimes 1_{\mathcal{B}}, \mathscr{D}[\mathcal{E}]\}$	✓	[illegible]	[illegible]	[illegible]	✓	✓	✓	$\lvert w_k\rangle\langle w_l\rvert \mapsto (q_k + q_l)^{-1}\{\rho, \mathcal{E}^{*}(\lvert w_k\rangle\langle w_l\rvert)\}$
Right bloom (13)	$(\rho \otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}]$ (e.g., two-state)	✗	✗	✗	✓	✓	✓	✓	$B \mapsto \rho\mathcal{E}^{*}(\mathcal{E}(\rho)^{-1}B)$ (e.g., weak values)
Left bloom (15)	$\mathscr{D}[\mathcal{E}](\rho \otimes 1_{\mathcal{B}})$ (e.g., correlator)	✗	✗	✗	✓	✓	✓	✓	$B \mapsto \mathcal{E}^{*}(B\mathcal{E}(\rho)^{-1})\rho$

2. Non-causal temporal correlation as a resource

[S. H. Lie and H. Kwon, private communication (2025)]

Q : What is a quantum state ρ?

A1 : An object that encodes
the probability distribution $\mathrm{Tr}[\rho M_i]$ for any POVM $\{M_i\}$.



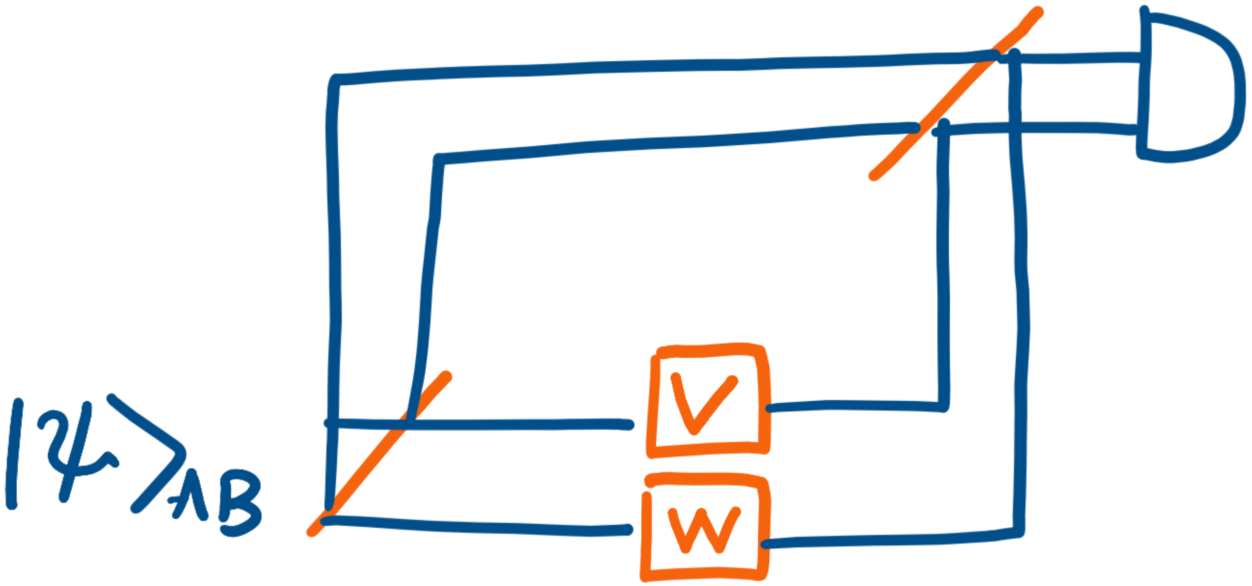
$$(|\psi\rangle_A|0\rangle_R + U|\psi\rangle_A|1\rangle_R)/\sqrt{2}$$

$$p(|\pm\rangle) = \frac{1 + \mathrm{Re}(\mathrm{Tr}[U\rho])}{2}$$

Q : What is a quantum state ρ?

A2 : An object that encodes
the interference term $\mathrm{Tr}[U\rho]$ for any unitary U.

It is also true for multipartite states



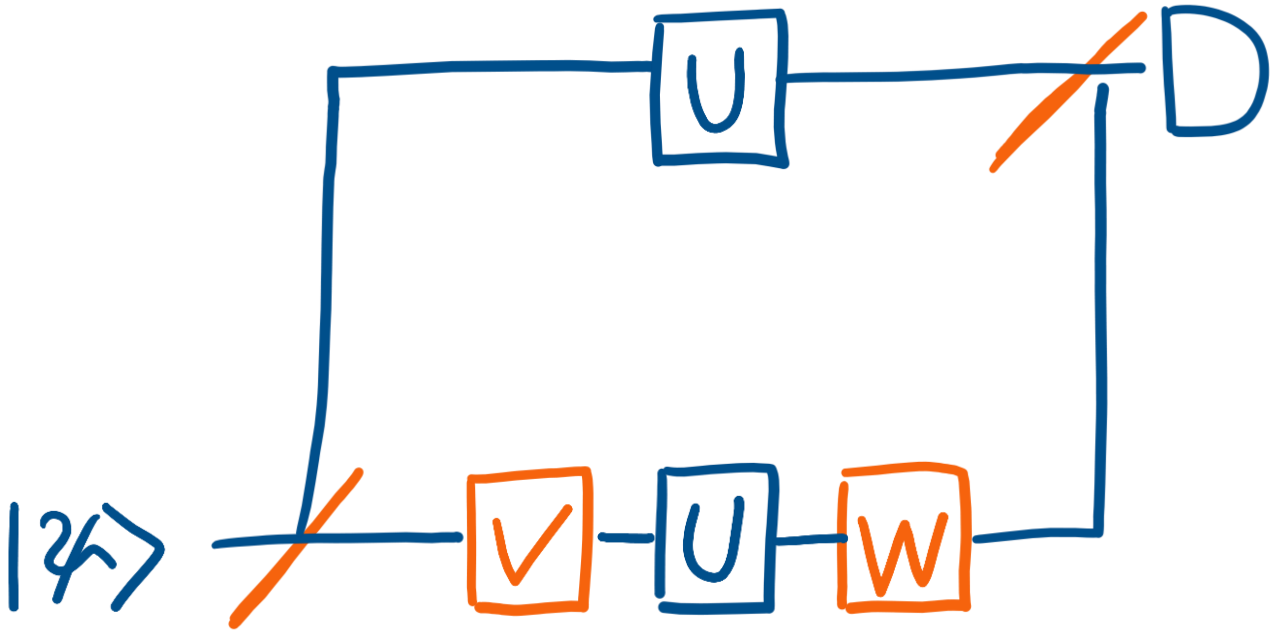
$$(|\psi\rangle_{AB}|0\rangle_R + V_A \otimes W_B|\psi\rangle_{AB}|1\rangle_R)/\sqrt{2}$$

$$p(|\pm\rangle) = \frac{1 + \mathrm{Re}(\mathrm{Tr}[V_A \otimes W_B \rho_{AB}])}{2}$$

Q : What is a quantum process?

A1 : A combination of an initial state ρ and a transformation U.

What about quantum processes?



$$(U|\psi\rangle_A|0\rangle_R + WUV|\psi\rangle_A|1\rangle_R)/\sqrt{2}$$

$$p(|\pm\rangle) = \frac{1 + \mathrm{Re}(\mathrm{Tr}[WUV\rho U^\dagger])}{2}$$

Q : What is a quantum process?

A2 : An object that encodes the **interference term** of the process interferometer!

- Let’s look into the interference term $\mathrm{Tr}\left[WUV\rho U^\dagger\right]$.

$$\mathrm{Tr}\left[WUV\rho U^\dagger\right] = \mathrm{Tr}\left[(V_A \otimes W_B)(\rho_A \otimes \mathbb{I}_B)D\left[\mathcal{U}_{B|A}\right]\right]$$
$$= \mathrm{Tr}\left[(V_A \otimes W_B)\mathcal{U}_{B|A} \star_R \rho_A\right]$$

- **The right-bloom naturally arises.**

CHEAT SHEET

$$\Phi_{B|A} \star_{\mathrm{FP}} \rho_A = \frac{1}{2}\left\{\rho_A \otimes I_B, D\left[\Phi_{B|A}\right]\right\}$$
$$\Phi_{B|A} \star_{R} \rho_A = (\rho_A \otimes I_B)\, D\left[\Phi_{B|A}\right]$$
$$\Phi_{B|A} \star_{\mathrm{L}} \rho_A = D\left[\Phi_{B|A}\right] (\rho_A \otimes I_B)$$

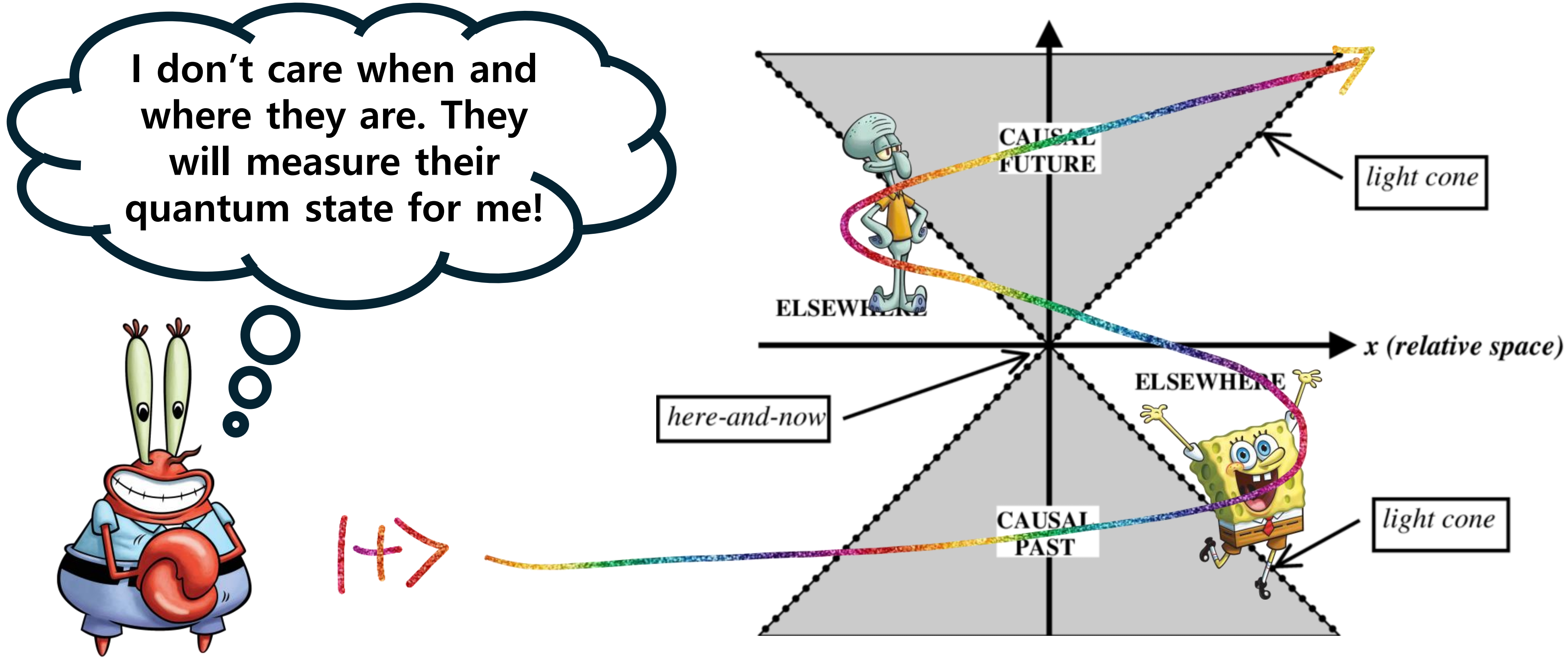
- This result generalizes to **general channels** $\mathcal{E}(\rho) = \mathrm{Tr}_\mathrm{E}\left[U(\rho_A \otimes |0\rangle\langle 0|_E)U^\dagger\right]$.

$$
\begin{gathered}
\langle\psi| \otimes \langle 0|(W \otimes \mathbb{I})U(V \otimes \mathbb{I})|\psi\rangle \otimes |0\rangle \\
\mathrm{Tr}_A\left[W\mathrm{Tr}_E[U\big(V\rho \otimes |0\rangle\langle 0|_E\big)U^\dagger]\right] \\
= \mathrm{Tr}[W\mathcal{E}(V\rho)] \\
= \mathrm{Tr}\left[(\mathbb{I}_A \otimes W_B)(V\rho_A \otimes \mathbb{I}_B)D\left[\mathcal{E}_{B|A}\right]\right] \\
= \mathrm{Tr}\left[(V_A \otimes W_B)\mathcal{E}_{B|A} \star_R \rho_A\right]
\end{gathered}
$$

- Therefore, you can indeed say that QSOTs are indeed quantum states; **they encode the interference term.** $(\rho_{AB} = \mathcal{E}_{B|A} \star_R \rho_A)$
- But why the right-bloom, not the FP function?

Spacetime black-box interferometry

- This kind of interferometric setting has the property of being **spacetime-agnostic**.
- You can just give out instructions to local parties to implement it **without knowing their spatiotemporal relation**.



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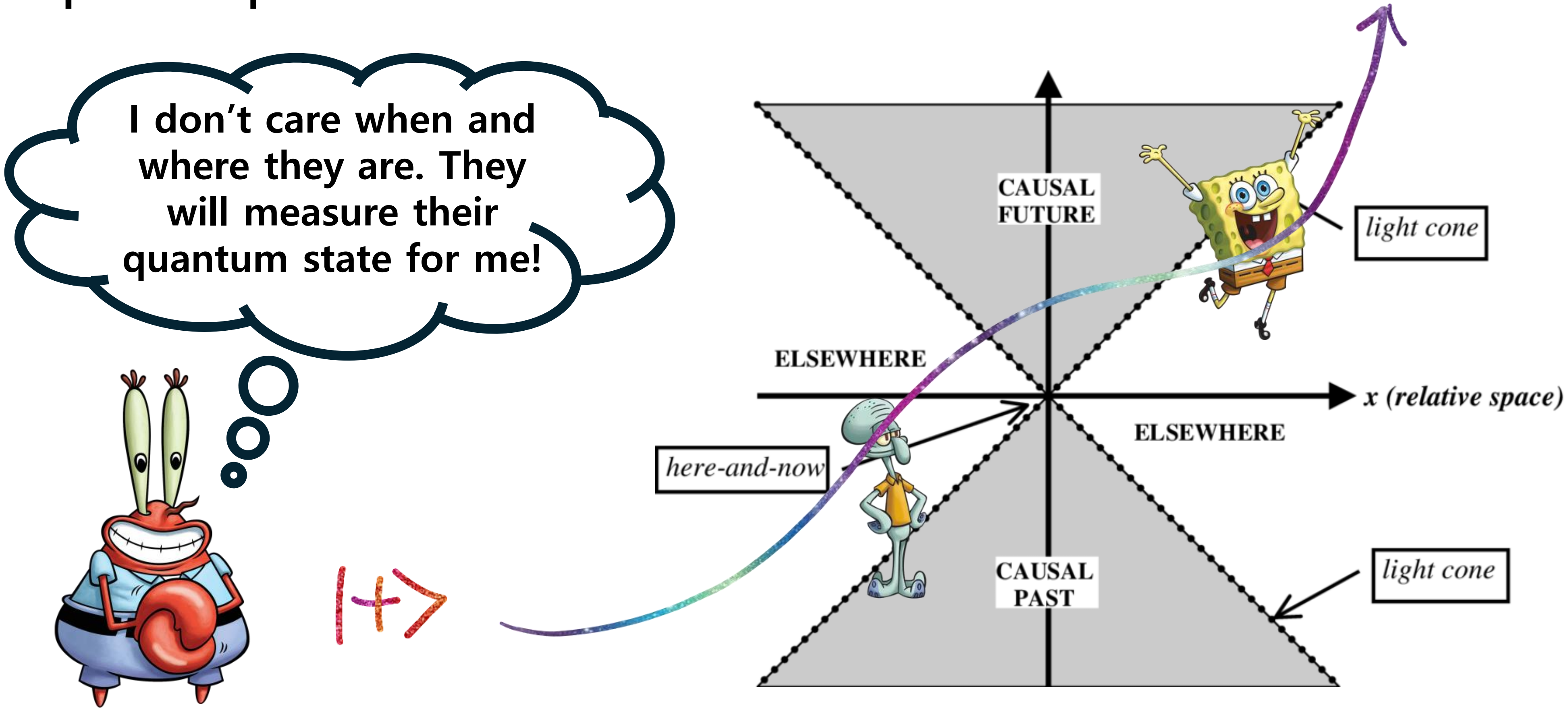
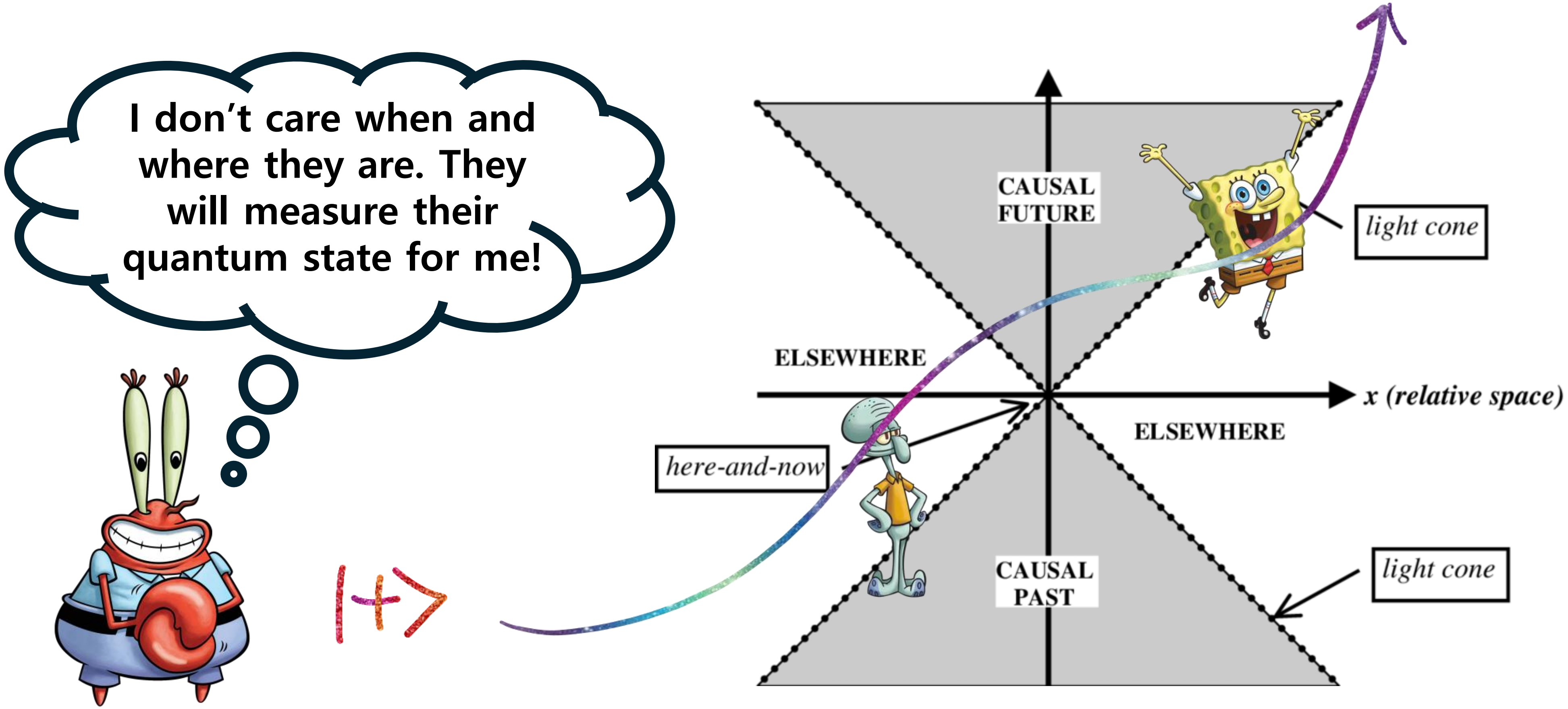


Spacetime black-box interferometry



- Nevertheless, instructions are **executable** and if there are multiple executable ways, then the measurement outcome is consistent results **regardless of their spatiotemporal relation**.

Temporal symmetry

- When we say that a quantum state $|\boldsymbol{\psi}\rangle$ **at** $\boldsymbol{t = t_a}$ **evolves into** $\boldsymbol{U|\psi\rangle}$ $\boldsymbol{t = t_b}$ **through a process** $\boldsymbol{U}$, we are making a significant assumption that the past and the future are fundamentally distinguishable.

- **However, who said** $\boldsymbol{t_b > t_a}$**?**

- A quantum state $\boldsymbol{U|\psi\rangle}$ **at** $\boldsymbol{t = t_b}$ **evolving into** $|\boldsymbol{\psi}\rangle$ **at** $\boldsymbol{t = t_a}$ **through** $\boldsymbol{U^\dagger}$ is also a perfectly fine description of the same dynamics.

- Notice it is different from saying that you cannot distinguish A from B.

Temporal symmetry

- Hence, without reference systems, our prediction of the interference term **cannot prefer one temporal direction over the other.**

- It should be the **even mixture** of
$\mathrm{Tr}\left[(V_A \otimes W_B)(\mathcal{U}_{B|A} \star_L \rho_A)\right]$ and $\mathrm{Tr}[(V_A \otimes W_B)\left(\mathcal{U}^{\dagger}_{A|B} \star_R U\rho U^{\dagger}_B\right)]$.

- Note that $\mathcal{U}_{B|A} \star_L \rho_A = \mathcal{U}^{\dagger}_{A|B} \star_R U\rho U^{\dagger}_B$.

- Hence the interference term takes the form of $\mathrm{Tr}\left[(V_A \otimes W_B)(\mathcal{U}_{B|A} \star_{FP} \rho_A)\right]$!

Temporal asymmetry

- But IRL we can easily tell past from future easily and access to the right-bloom.

- How is this possible?

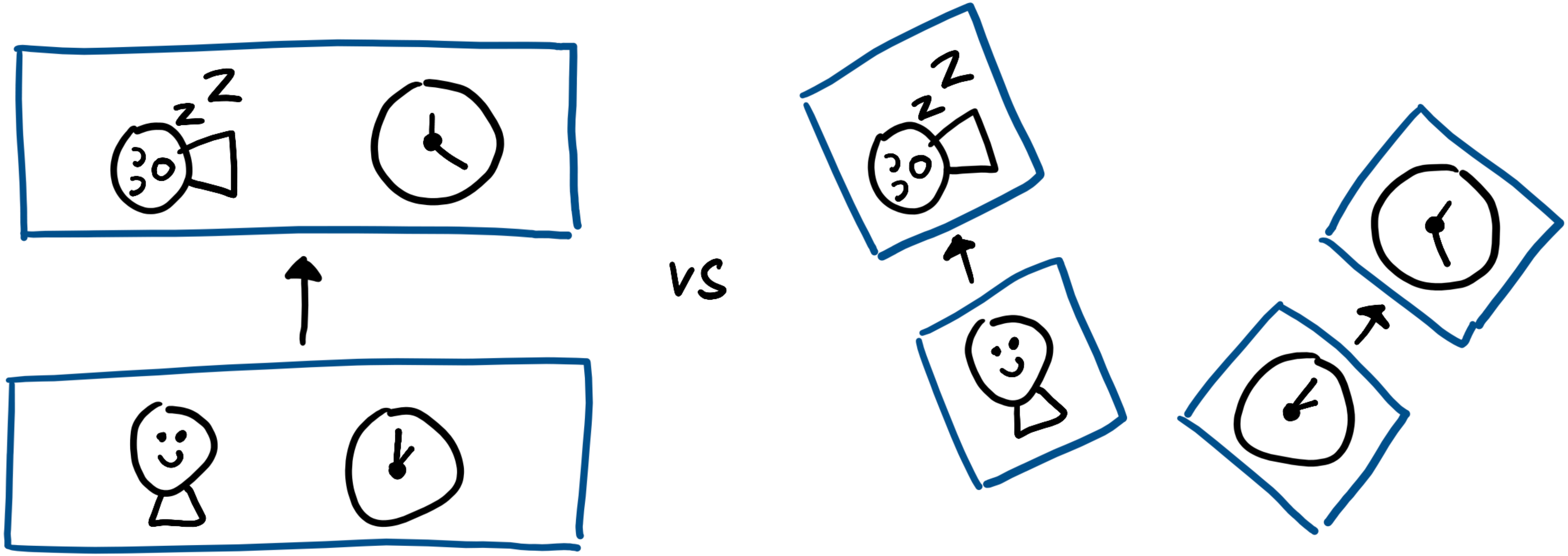
Temporal asymmetry

- But IRL we can easily tell past from future easily and access to the right-bloom.
- How is this possible?
- **Clocks!**
- A clock actually has two roles:

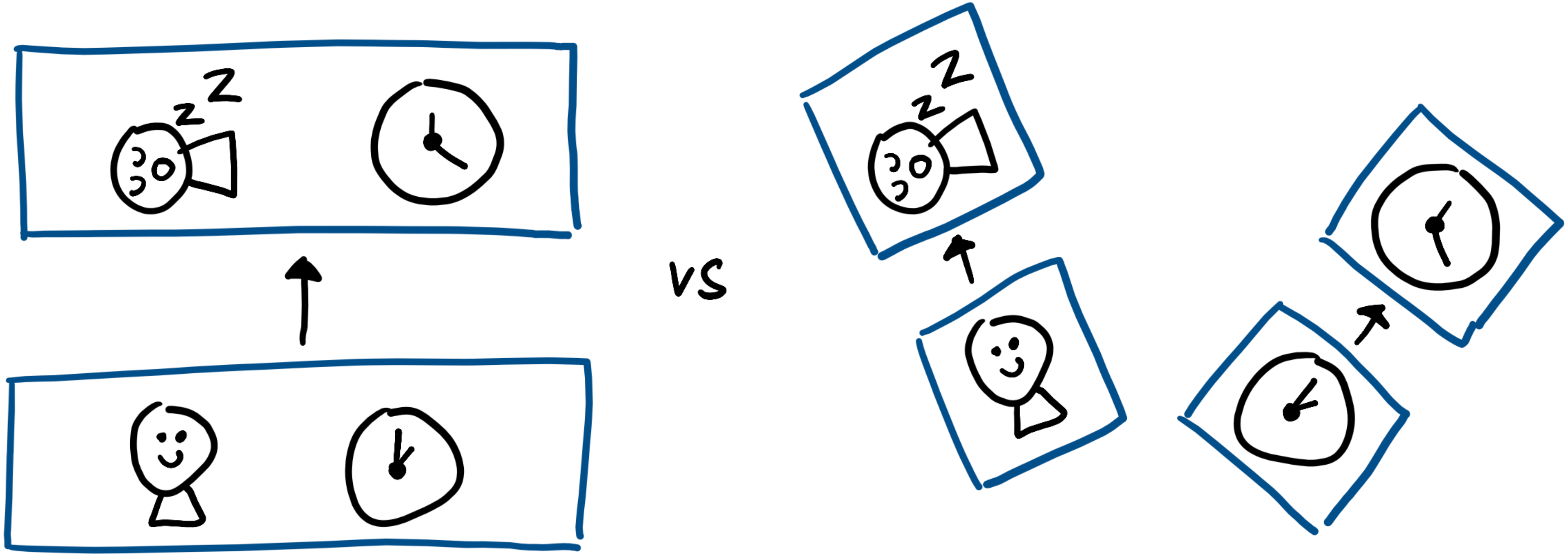
(1) It tells us **what time it is** (time-map)
(2) It tells us **into which direction the time flows**. (time-compass)

Clocks?

- Recall how clocks work.
- A clock and another system, although they do not really interact, appear to have **SOME correlation**.



Clocks?



- But it **cannot be captured** by the conventional quantum state; they stay in a **product state** $\rho_A \otimes \sigma_C$ at all time.

Axioms for retrodiction: achieving time-reversal symmetry with a prior

Arthur J. Parzygnat and Francesco Buscemi

2023-05-12

Tensoriality is not desirable

we expect retrodiction to satisfy *tensoriality*, which states

$$\mathscr{R}_{\alpha,\mathcal{E}} \otimes \mathscr{R}_{\alpha',\mathcal{E}'} = \mathscr{R}_{\alpha\otimes\alpha',\mathcal{E}\otimes\mathcal{E}'}.$$

- This axiom sounds convincing at first, but in terms of QSOTs, it amounts to saying

$$(\mathcal{E}_A \otimes \mathcal{F}_B) \star (\rho_A \otimes \sigma_B) = (\mathcal{E}_A \star \rho_A) \otimes (\mathcal{F} \star \sigma_B)$$

- However, the information that A and B were prepared in ρ_A and σ_B **simultaneously** and evolve in parallel is a data worth attention!

Tensoriality is not desirable

- The FP function captures this correlation, i.e., in general,

$$(\mathcal{E}_A \otimes \mathcal{F}_B) \star_{FP} (\rho_A \otimes \sigma_B) \neq (\mathcal{E}_A \star_{FP} \rho_A) \otimes (\mathcal{F} \star_{FP} \sigma_B)$$

A clock as a resource

- Let's assume that now you append a '**clock system**' C_A that evolves into C_B sitting next to your dynamics between A and B.

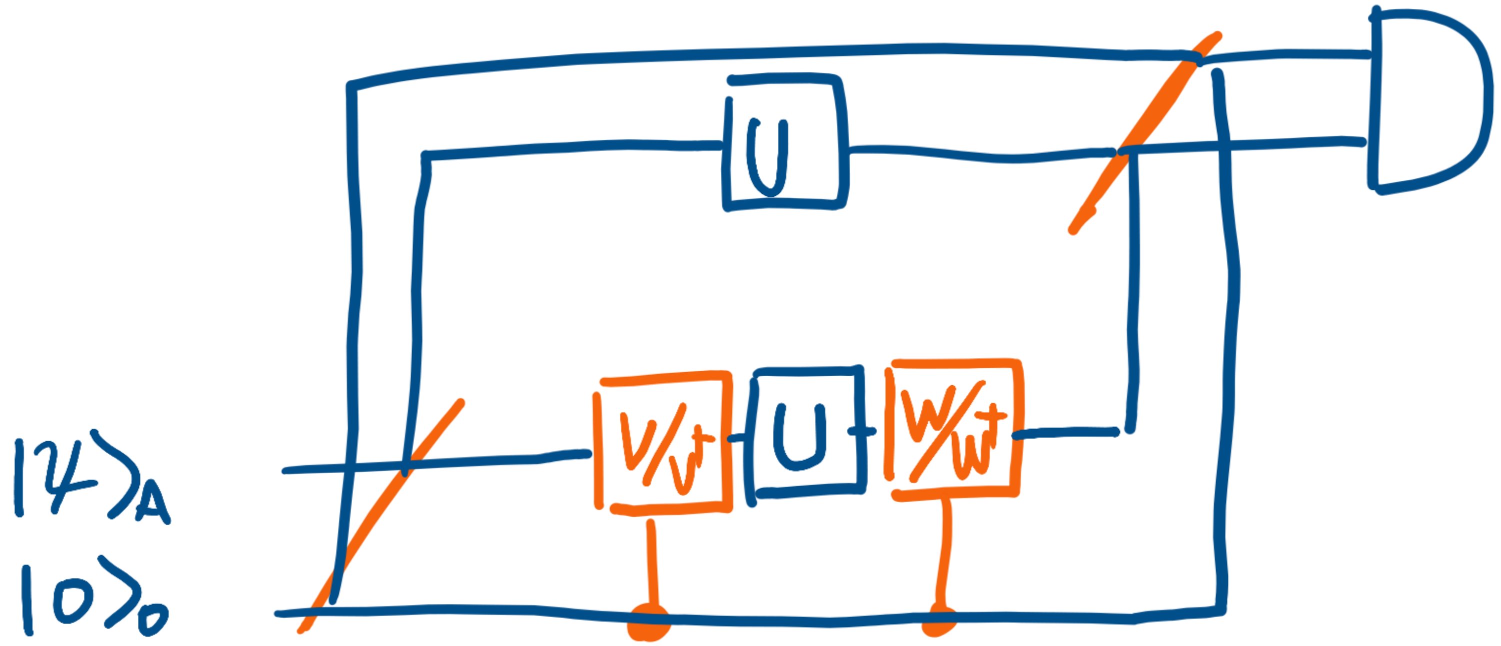
- It does **nothing**; it is initialized in $|0\rangle_{C_A}$ and just stays there.

- The corresponding QSOT is

$$\left(\mathcal{E}_{B|A} \otimes \mathrm{id}_{C_B|C_A}\right) \star \left(\rho_A \otimes |0\rangle\langle 0|_{C_A}\right)$$

- Now, what can we use this clock for?

Process interferometry with clock



- Now we consider the same interferometry, but with the clock included.

- Consider $\tilde{V}_{AC_A} = V_A \otimes |1\rangle\langle 0|_{C_A} + V_A^\dagger \otimes |0\rangle\langle 1|_{C_A}$
  and $\widetilde{W}_{BC_B} = W_A \otimes |0\rangle\langle 1|_{C_A} + V_A^\dagger \otimes |1\rangle\langle 0|_{C_A}$.

Process interferometry with clock

- Then the interference term becomes

$$\mathrm{Tr}\left[\left(\tilde{V}_{AC_A} \otimes \widetilde{W}_{BC_B}\right)\left(\mathcal{E}_{B|A} \otimes \mathrm{id}_{C_B|C_A}\right) \star_{FP} (\rho_A \otimes |0\rangle\langle 0|_{C_A})\right]$$
$$= \mathrm{Re}\left(\mathrm{Tr}\left[(V_A \otimes W_B)\left(\mathcal{E}_{B|A} \star_R \rho_A\right)\right]\right)$$

- By varying $V \to e^{i\theta}V$, one can access the $\mathrm{Tr}\left[(V_A \otimes W_B)\left(\mathcal{E}_{B|A} \star_R \rho_A\right)\right]$!

- One can now access the **temporally asymmetric statistics with the help of a clock**.

POVMs over time

- Moreover, for the case of the FP function, the measurement probability of the interferometry

$$P = \frac{1 \pm \mathrm{Re}(\mathrm{Tr}[(V_A \otimes W_B)(\mathcal{E} \star_{FP} \rho)])}{2}$$

allows for the expression (because $(\mathcal{E} \star_{FP} \rho)$ is a Hermitian operator)

$$P = \frac{\mathrm{Tr}[(\mathbb{I}_{AB} \pm \mathrm{Re}(V \otimes W))(\mathcal{E} \star_{FP} \rho)]}{2}$$

- Here $\mathbb{I}_{AB} \pm \mathrm{Re}(V \otimes W)$ are always positive operators that sum up to $\mathbb{I}_{AB}$ for arbitrary unitary operators V_A and W_B.

- So, they can be interpreted as **a POVM element for QSOTs**!

Clock-correlation

- Physical systems propagating into the same direction in time definitely **have correlation**.

- This cannot be a **spatial correlation**; it cannot be conventional quantum state over space.

- This is also not a **causation**; neither clock nor you watching it is a cause of the other.

- This demonstrates that there exists **noncausal temporal correlation**, and it is indeed a resource for telling time.

- We couldn't have captured this correlation with the conventional formalism;
  The QSOT formalism is a useful tool for analyzing the **clock correlation**.

- Maybe we can use the framework for constructing a resource theory of **dynamical resources.**

Thank you for listening!