A solution of the generalised quantum Stein's lemma

IEEE Trans. Inf. Theory 2025 (arXiv:2408.06410)

Ludovico Lami

University of Amsterdam &
Scuola Normale Superiore, Pisa, Italy

University of Amsterdam

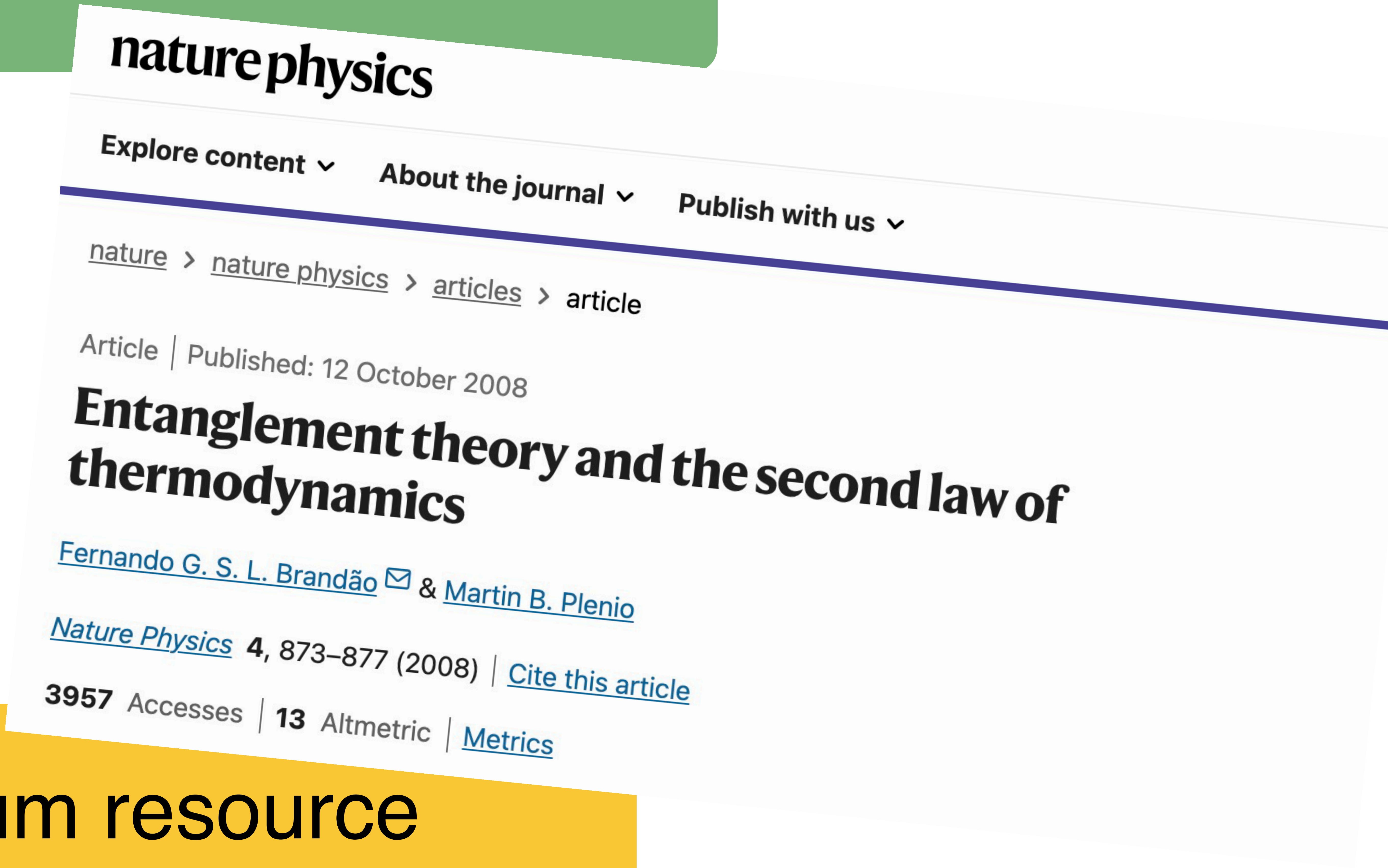
QuSoft
Research Center for Quantum Software

Scuola Normale Superiore

Quantum hypothesis testing

Quantum resource manipulation

Quantum hypothesis testing


nature physics

Explore content ⌄ About the journal ⌄ Publish with us ⌄

nature › nature physics › articles › article

Article | Published: 12 October 2008

Entanglement theory and the second law of thermodynamics

Fernando G. S. L. Brandão & Martin B. Plenio

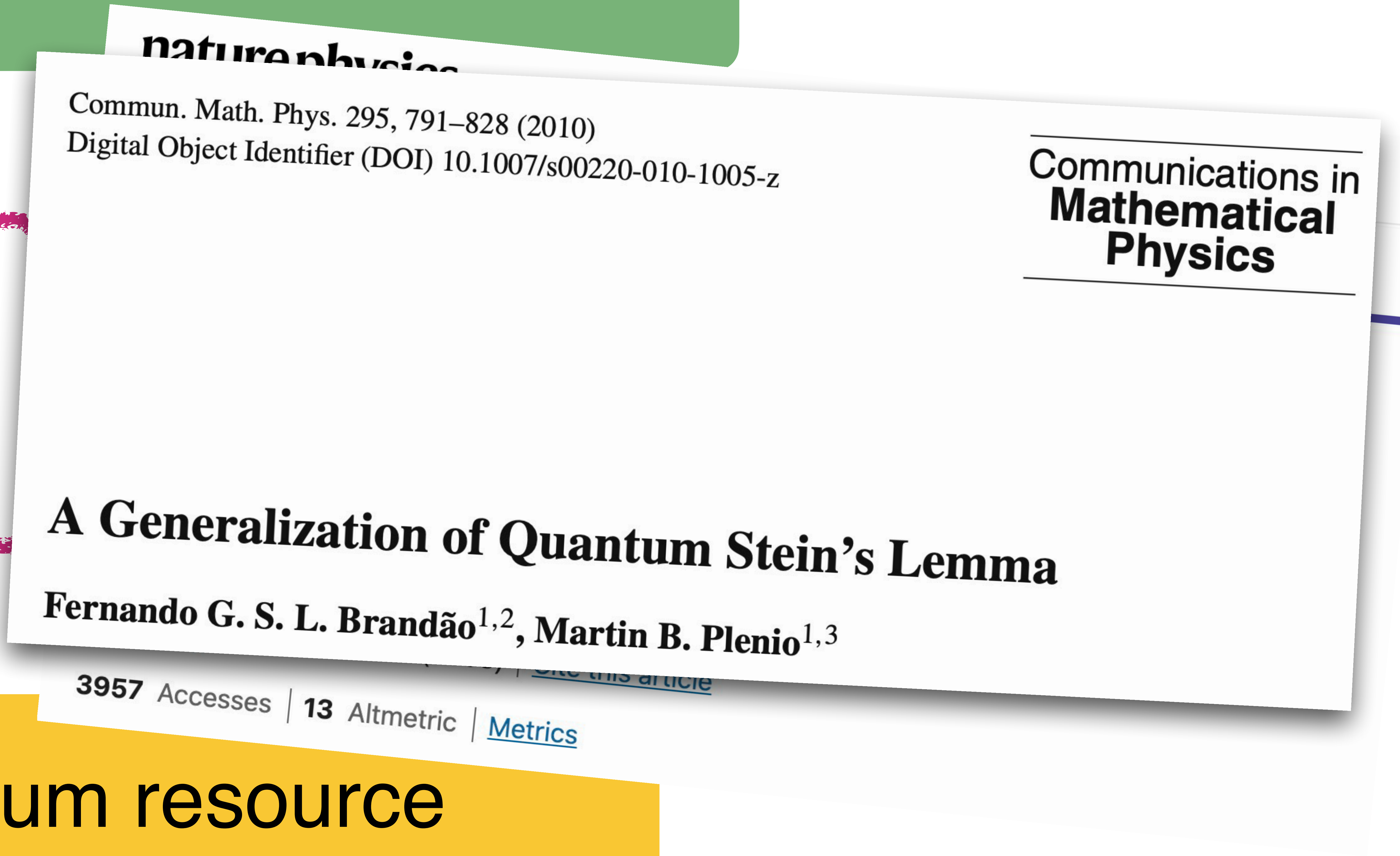
Nature Physics **4**, 873–877 (2008) | Cite this article

Quantum resource manipulation

Quantum hypothesis testing

Generalised quantum Stein's lemma

Quantum resource manipulation


nature physics

Commun. Math. Phys. 295, 791–828 (2010)
Digital Object Identifier (DOI) 10.1007/s00220-010-1005-z

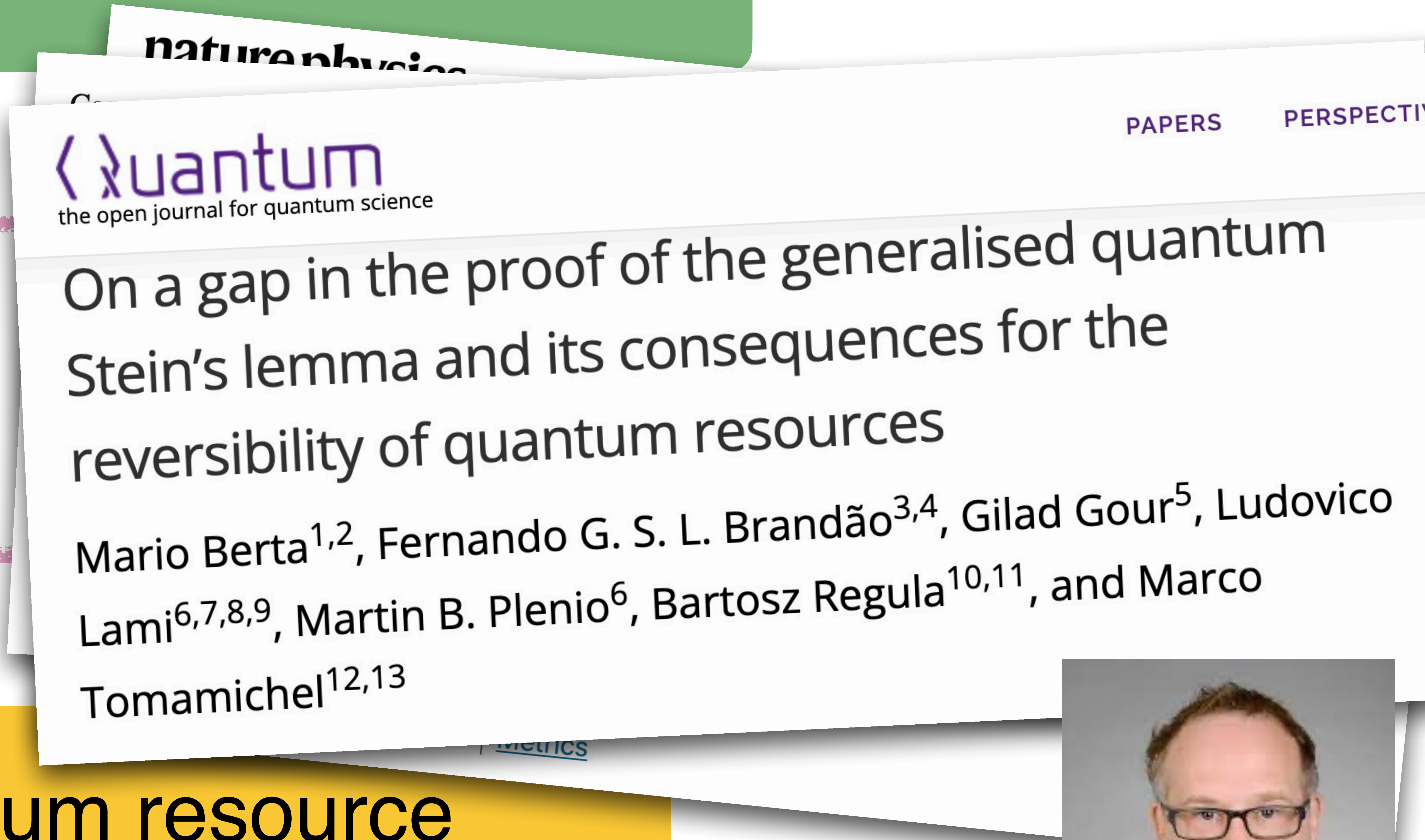
Communications in Mathematical Physics

A Generalization of Quantum Stein's Lemma

Fernando G. S. L. Brandão[1,2], Martin B. Plenio[1,3]

Quantum hypothesis testing

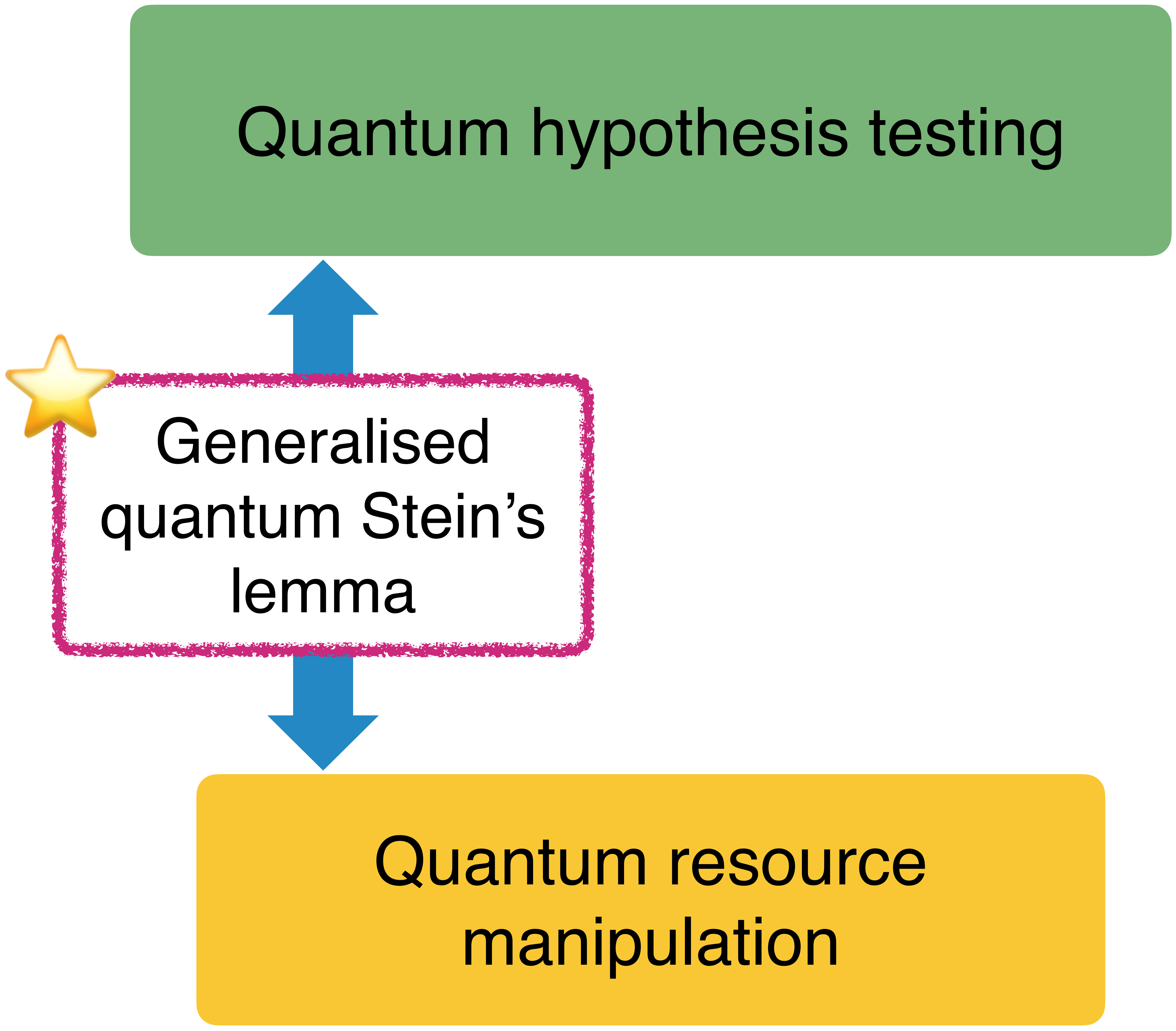
Generalized quantum Stein's lemma

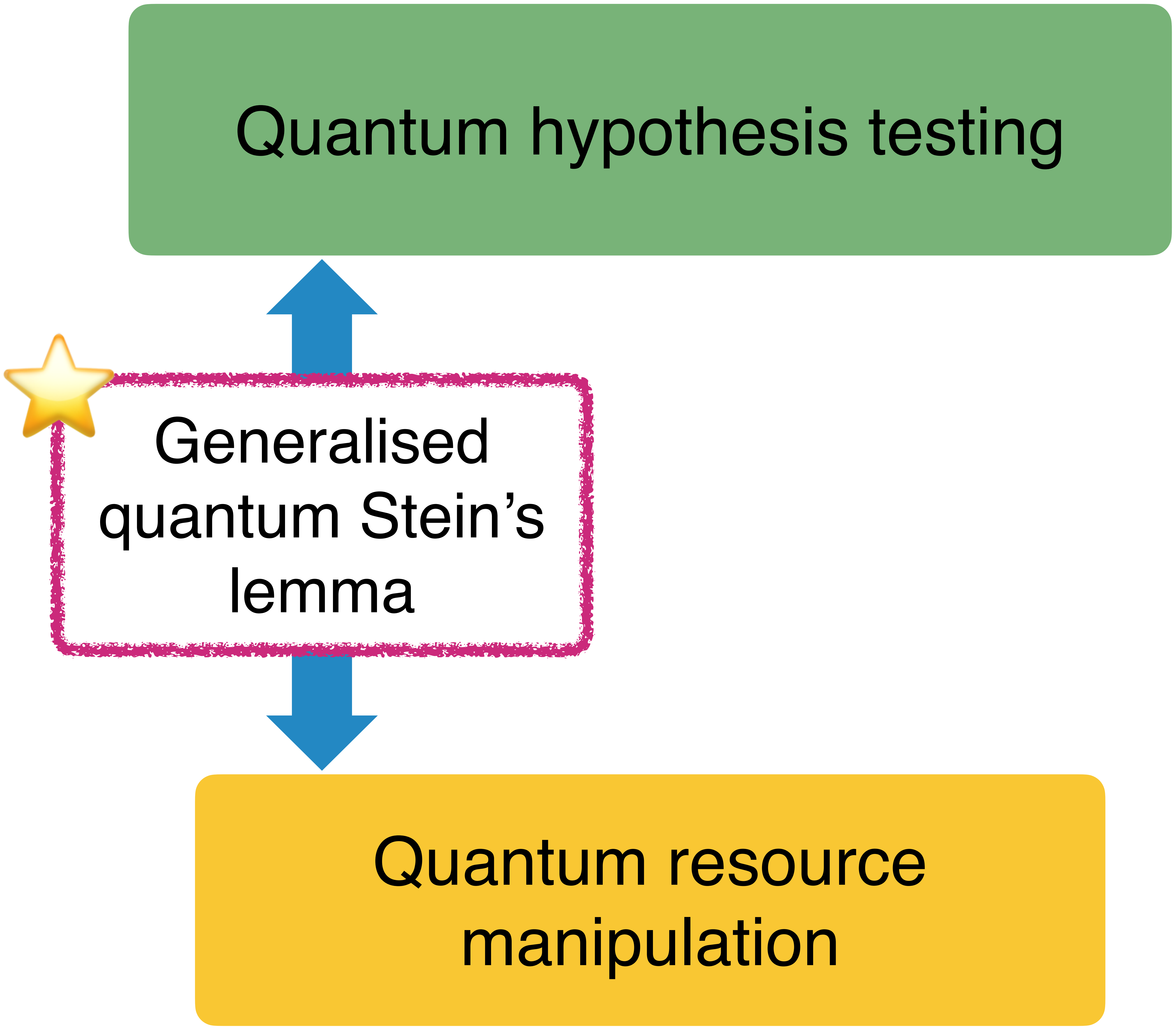

Quantum
the open journal for quantum science
PAPERS PERSPECTI

On a gap in the proof of the generalised quantum Stein's lemma and its consequences for the reversibility of quantum resources

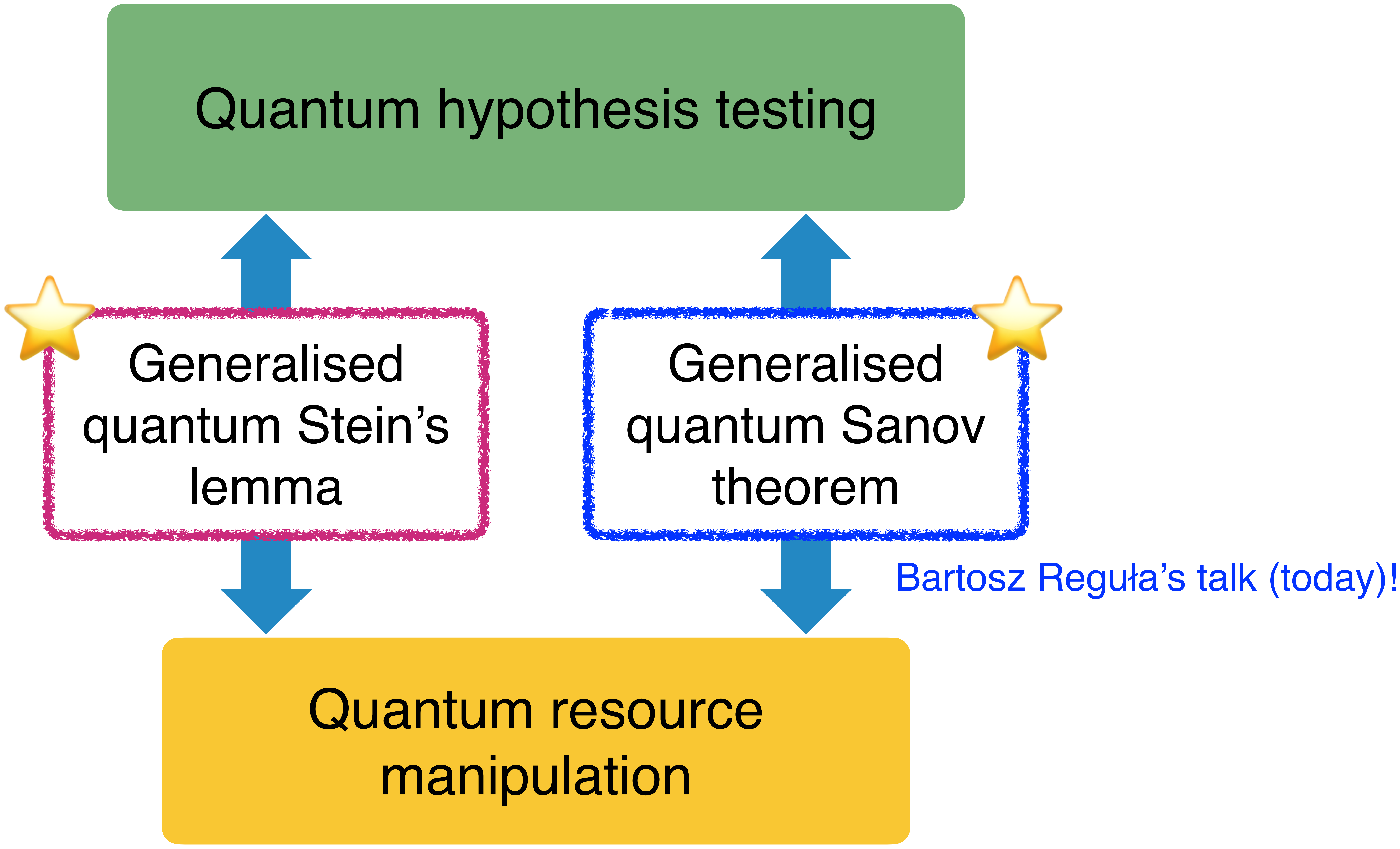
Mario Berta[1,2], Fernando G. S. L. Brandão[3,4], Gilad Gour[5], Ludovico Lami[6,7,8,9], Martin B. Plenio[6], Bartosz Regula[10,11], and Marco Tomamichel[12,13]

Quantum resource manipulation

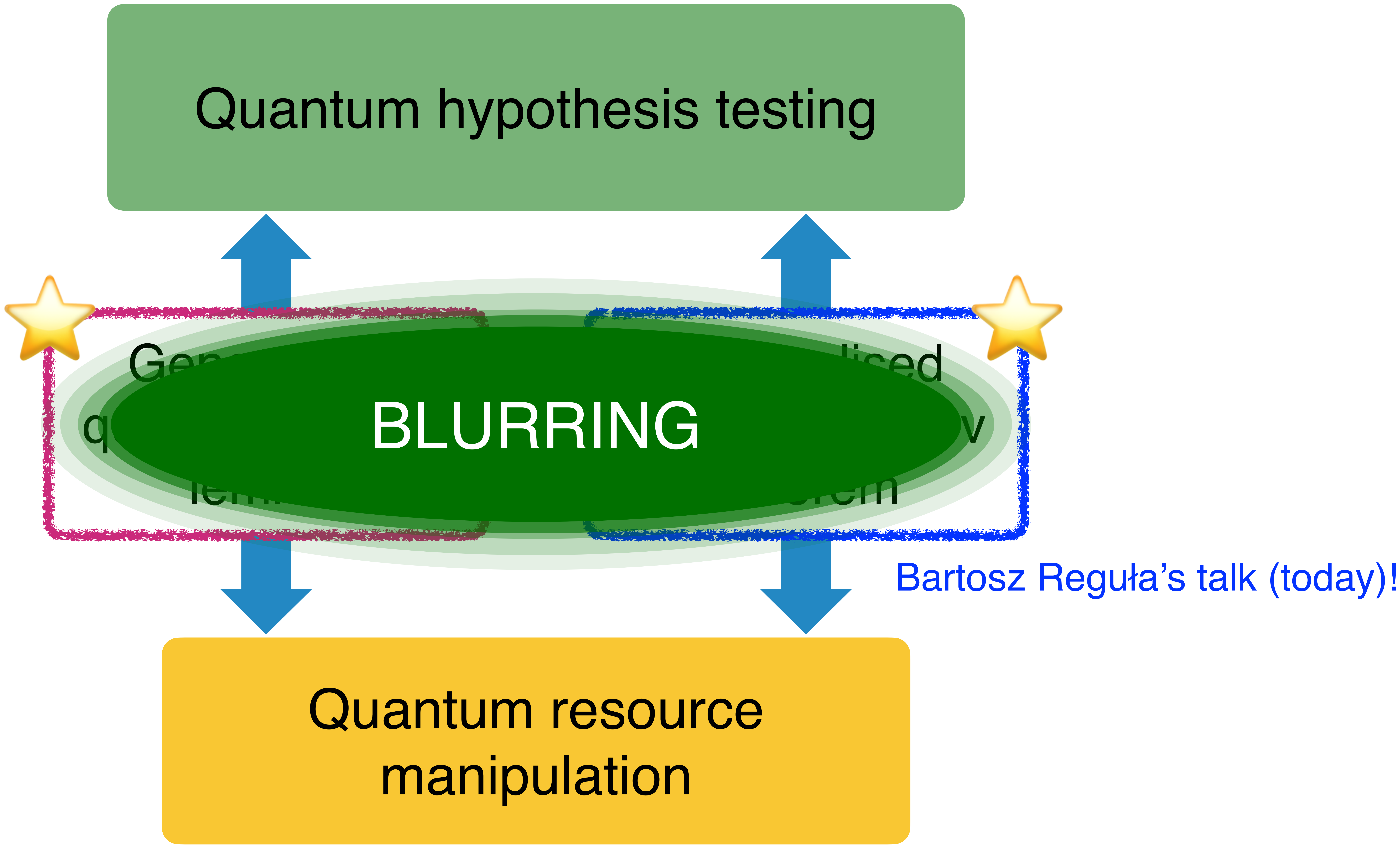

Quantum hypothesis testing
Generalised quantum Stein's lemma
Quantum resource manipulation



See also Masahito Hayashi's talk (tomorrow)



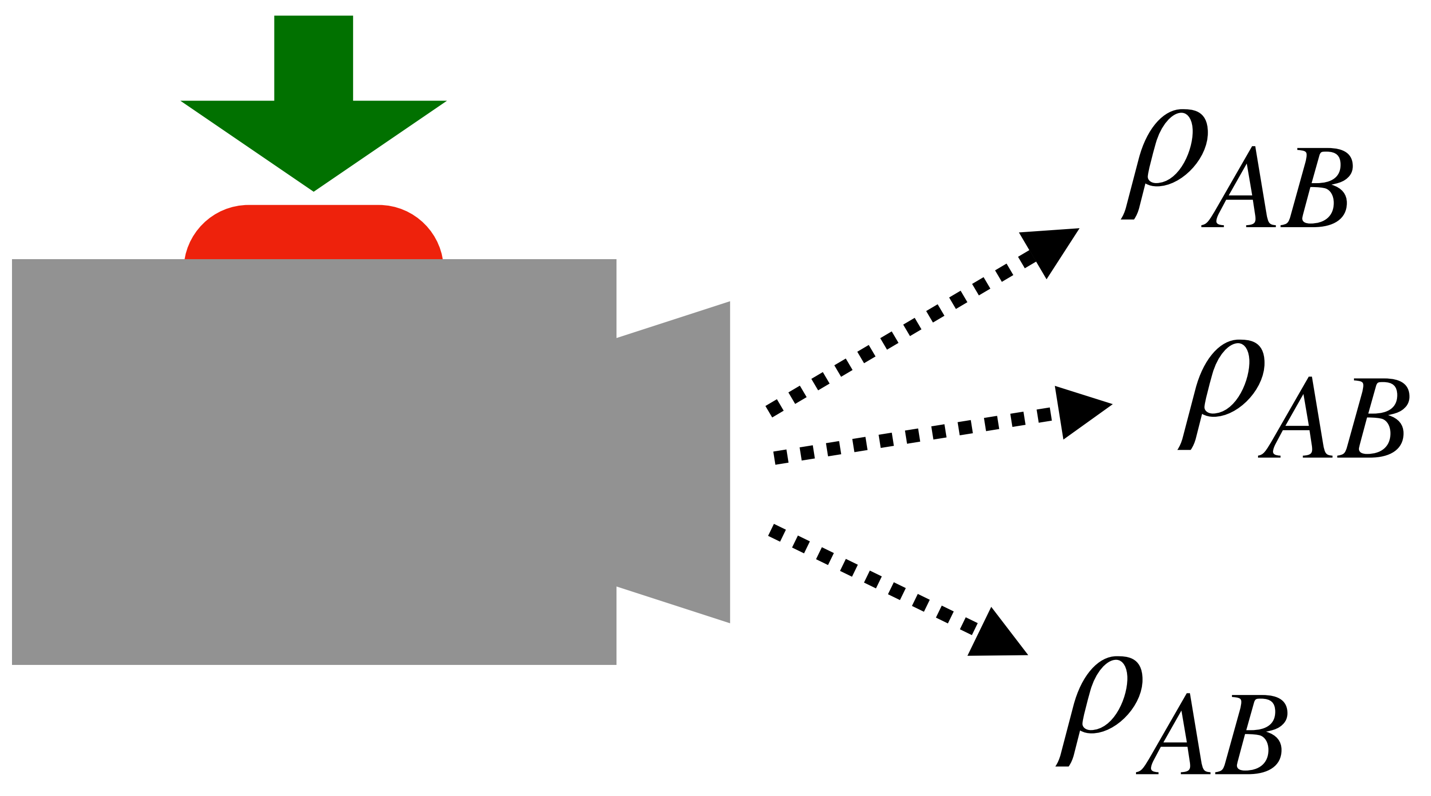
See also Masahito Hayashi’s talk (tomorrow)



See also Masahito Hayashi’s talk (tomorrow)

Entanglement testing

- You bought a device that should produce on demand a known **entangled** state ρ_{AB}.



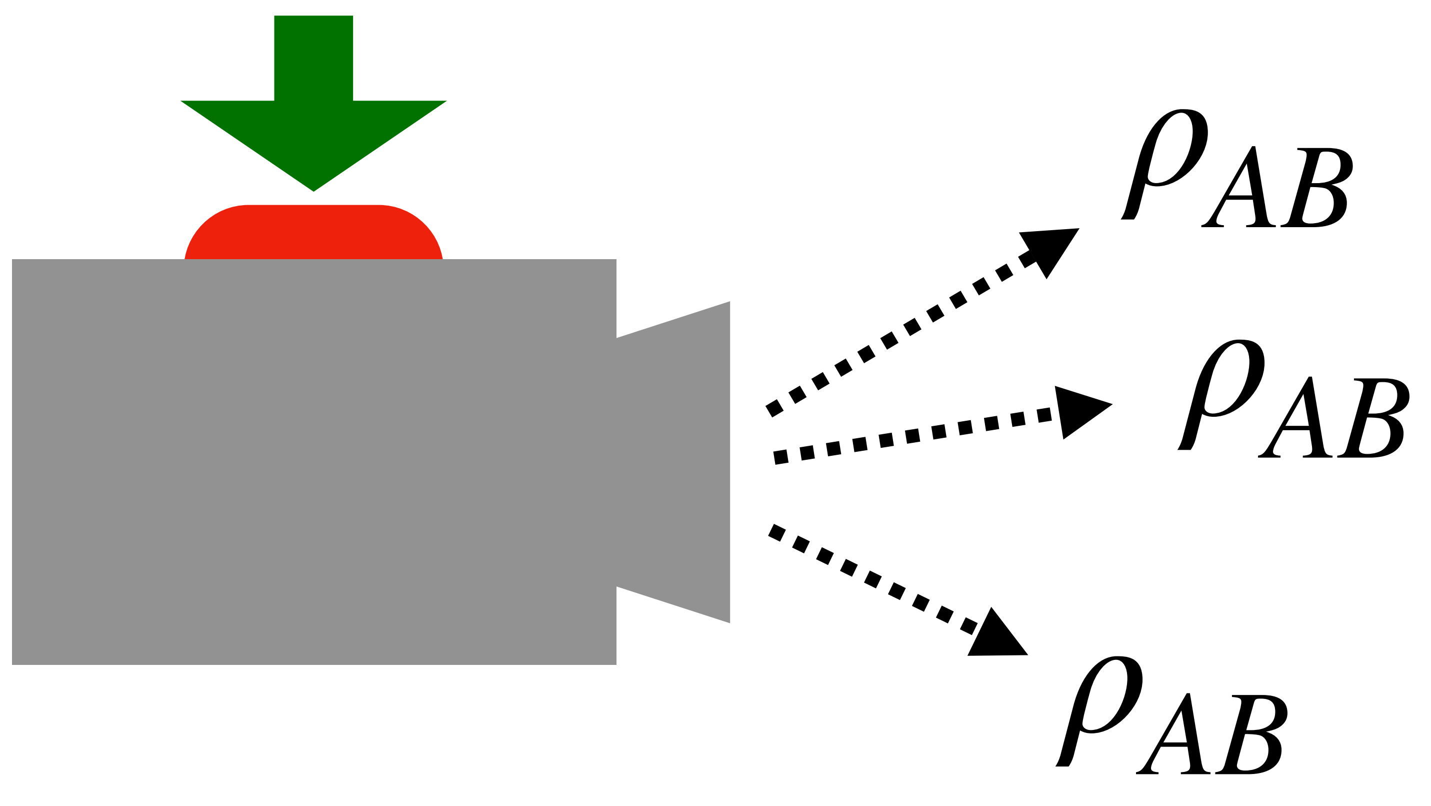
Entangled ≔ non-separable. Separable states:

$$\mathcal{S}_{A:B} = \operatorname{conv}\left\{ |\alpha\rangle\langle\alpha|_A \otimes |\beta\rangle\langle\beta|_B : \ |\alpha\rangle_A \in \mathcal{H}_A, |\beta\rangle_B \in \mathcal{H}_B, \ \| |\alpha\rangle_A \| = \| |\beta\rangle_B \| = 1 \right\}$$

Entanglement testing

- You bought a device that should produce on demand a known **entangled** state ρ_{AB}.

- Perhaps the device is maliciously engineered to output a global **separable** state even after *n* uses.



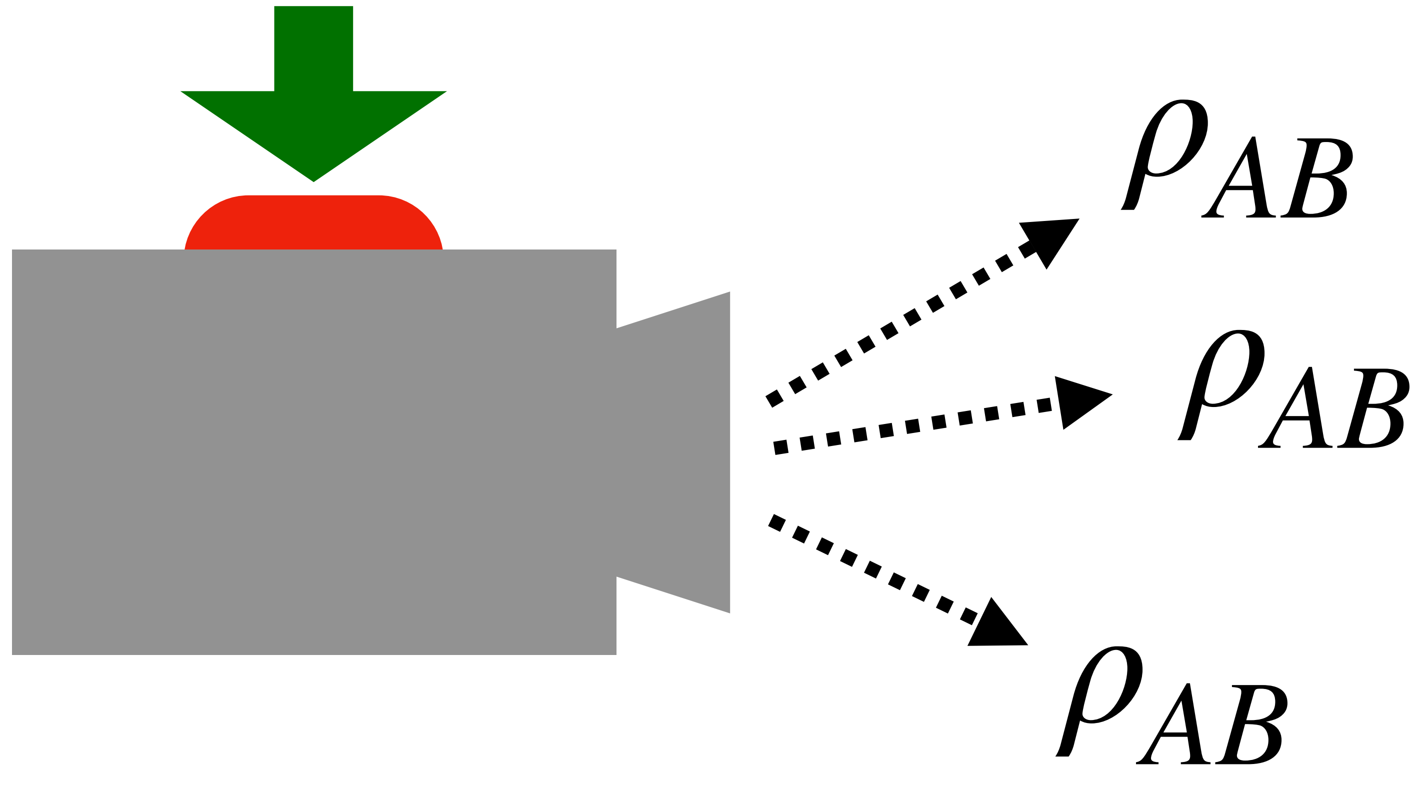
Entangled ≔ non-separable. Separable states:

$$\mathcal{S}_{A:B} = \text{conv}\left\{ |\alpha\rangle\langle\alpha|_A \otimes |\beta\rangle\langle\beta|_B : \ |\alpha\rangle_A \in \mathcal{H}_A, \ |\beta\rangle_B \in \mathcal{H}_B, \ \||\alpha\rangle_A\| = \||\beta\rangle_B\| = 1 \right\}$$

Entanglement testing

- You bought a device that should produce on demand a known **entangled** state ρ_{AB}.

- Perhaps the device is maliciously engineered to output a global **separable** state even after *n* uses.

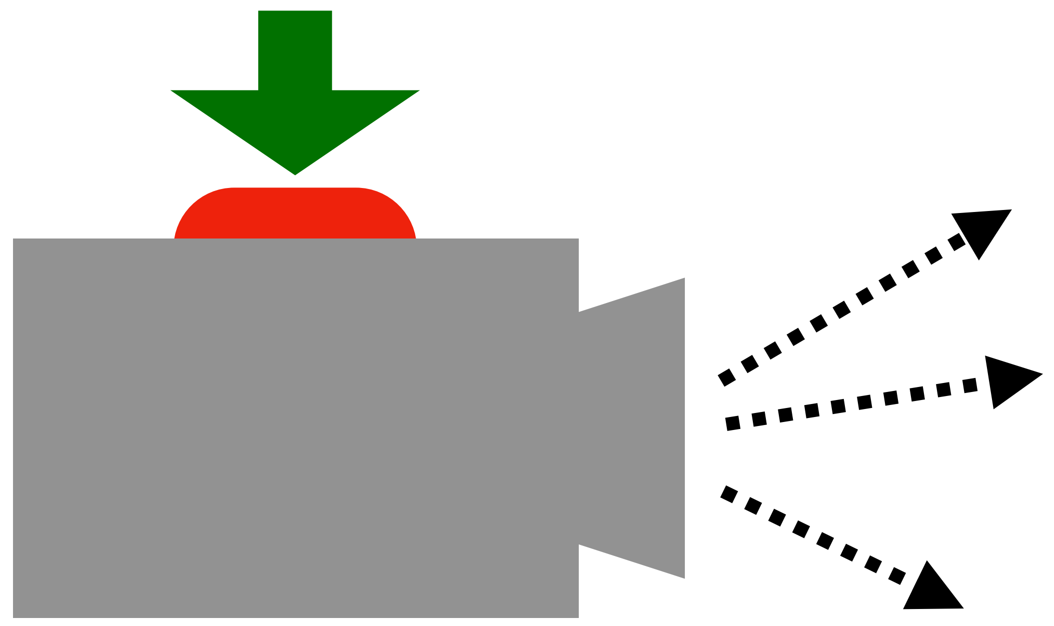


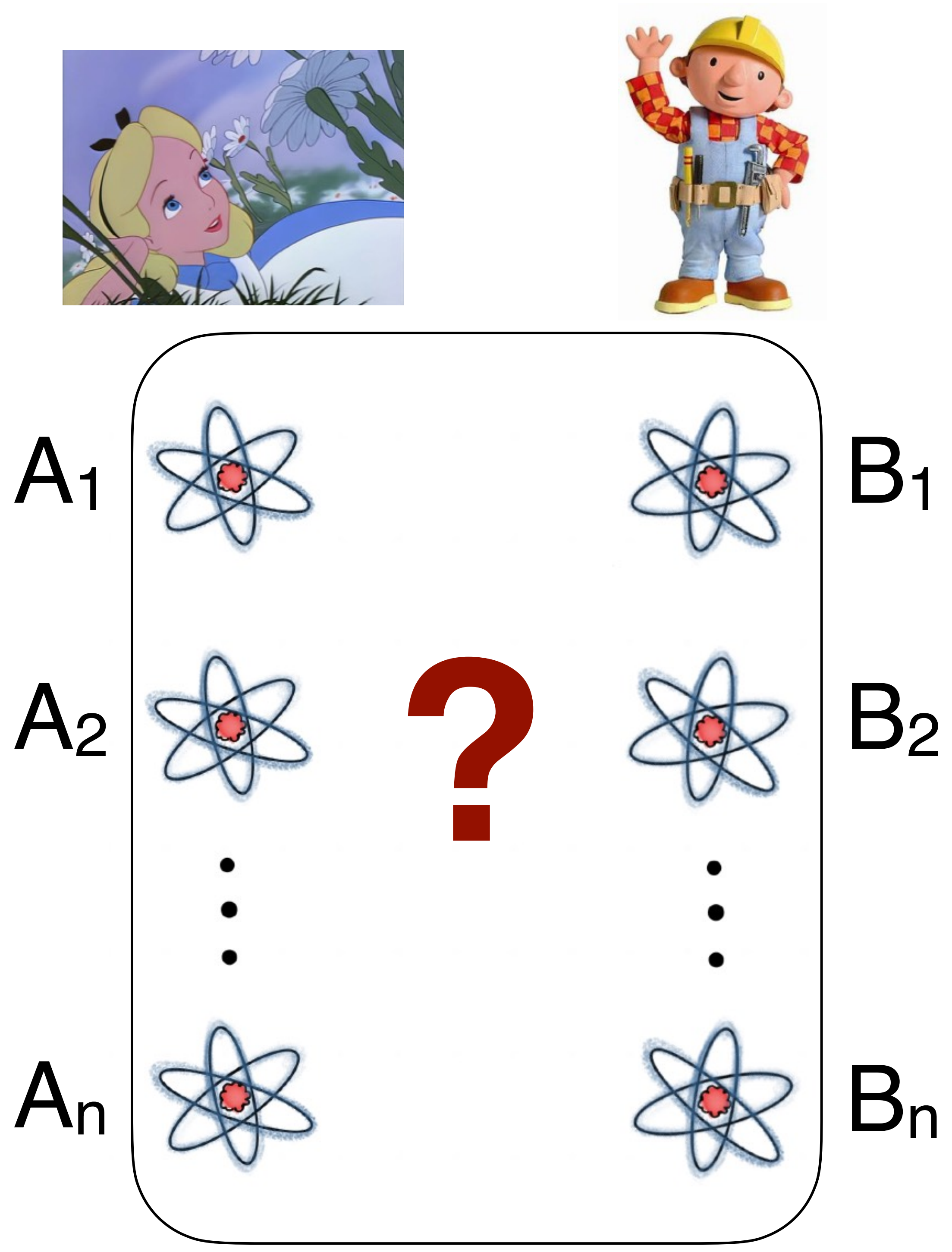
How effectively can you decide? (min. number of uses)

Entangled ≔ non-separable. Separable states:

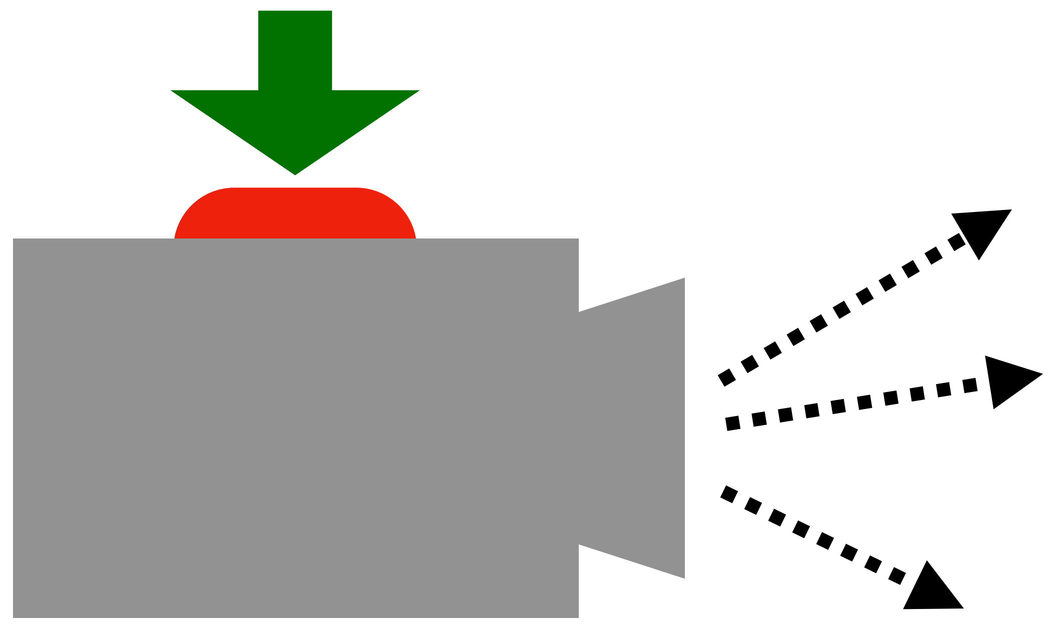
$$\mathcal{S}_{A:B} = \mathrm{conv}\left\{ |\alpha\rangle\langle\alpha|_A \otimes |\beta\rangle\langle\beta|_B : \ |\alpha\rangle_A \in \mathcal{H}_A,\ |\beta\rangle_B \in \mathcal{H}_B,\ \| |\alpha\rangle_A \| = \| |\beta\rangle_B \| = 1 \right\}$$

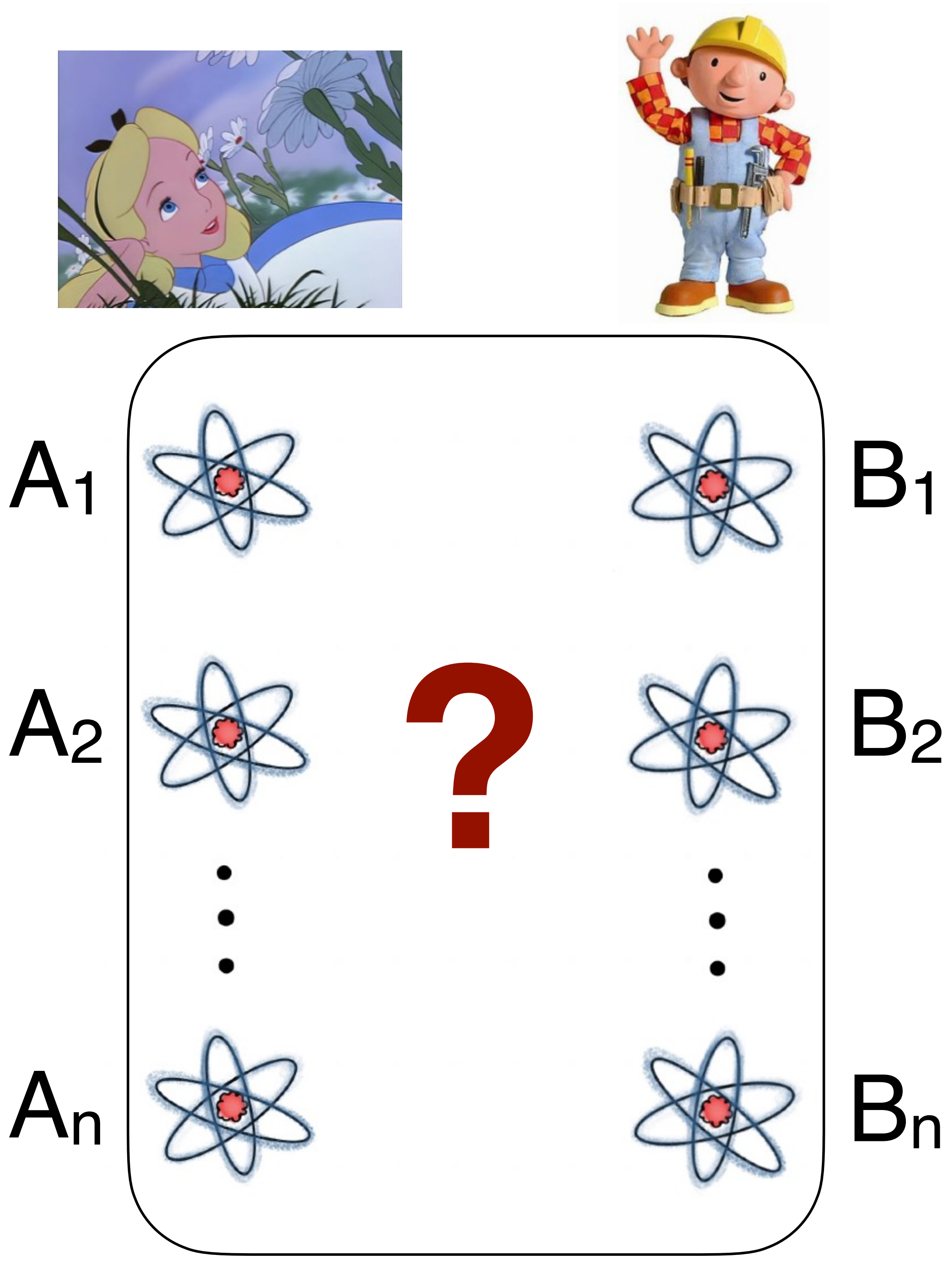
After *n* uses:



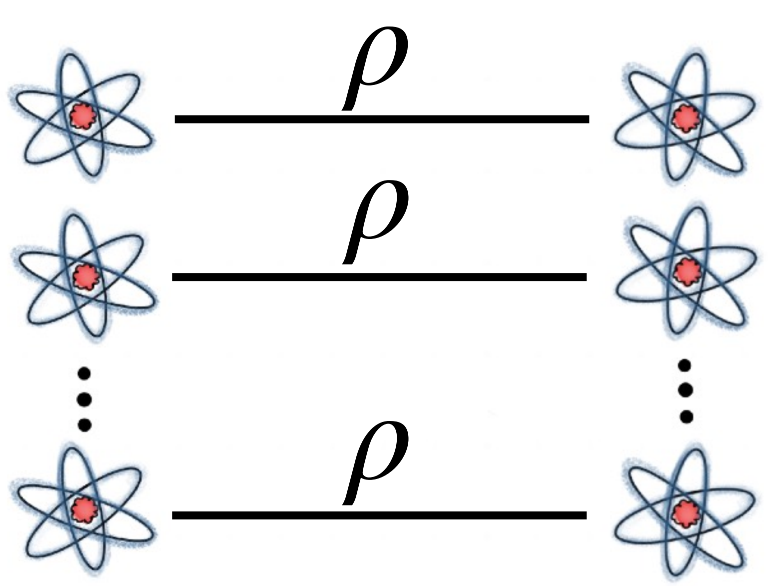


After n uses:



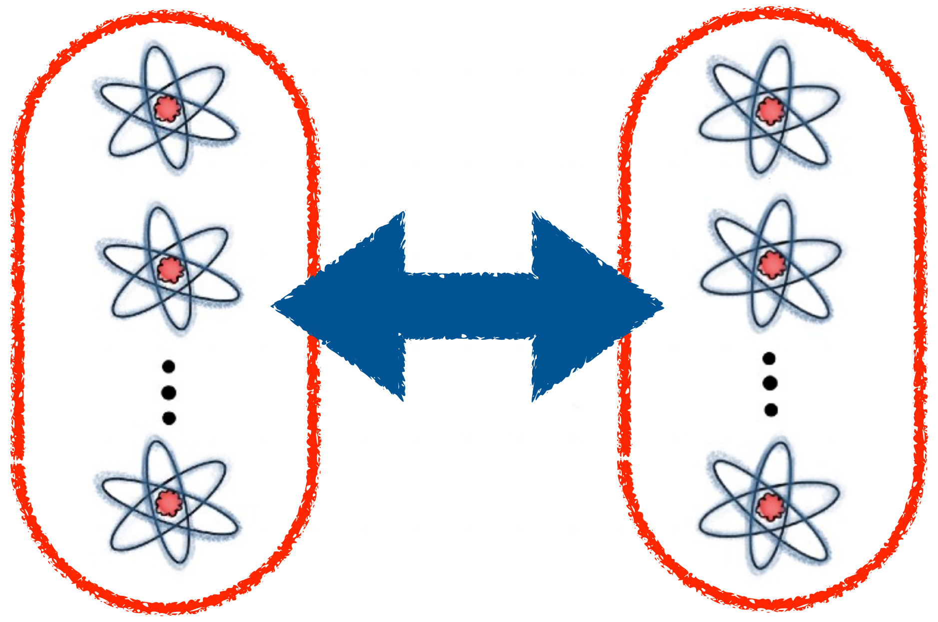


Hypothesis 1: $\rho_{AB}^{\otimes n}$

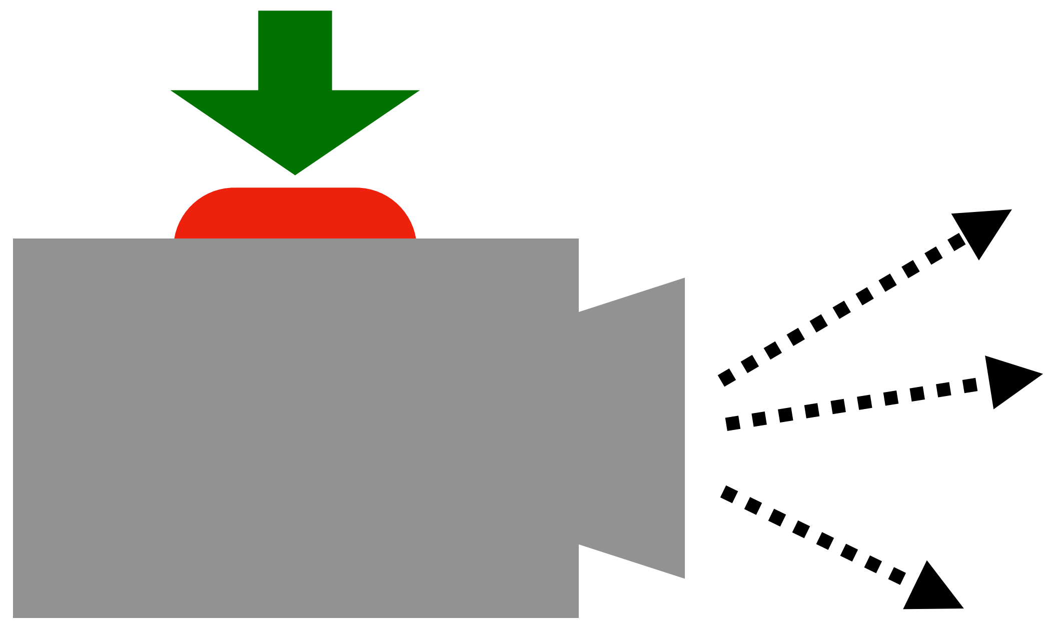


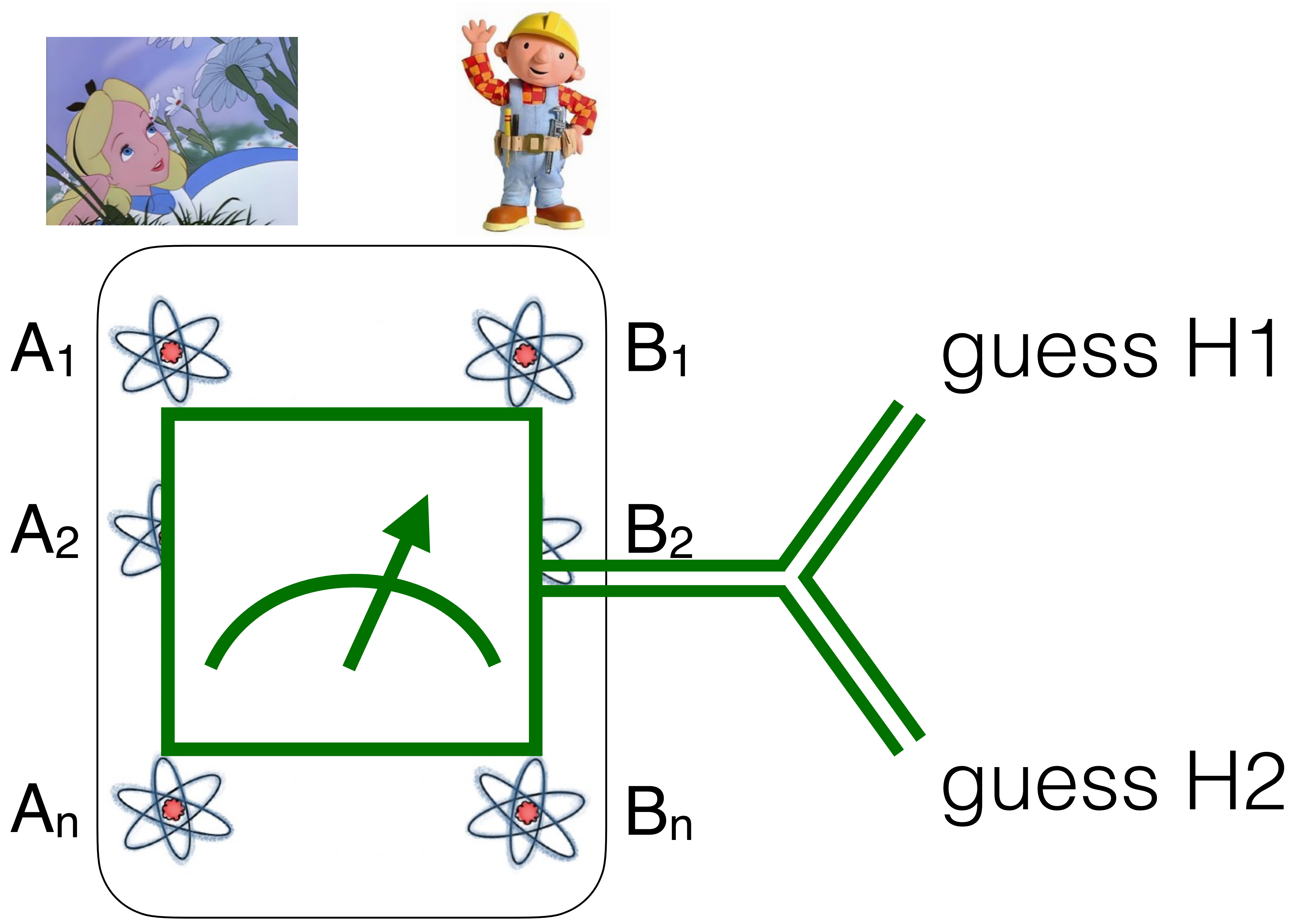
vs

Hypothesis 2: $\sigma_{A^nB^n} \in \mathcal{S}_{A^n:B^n}$

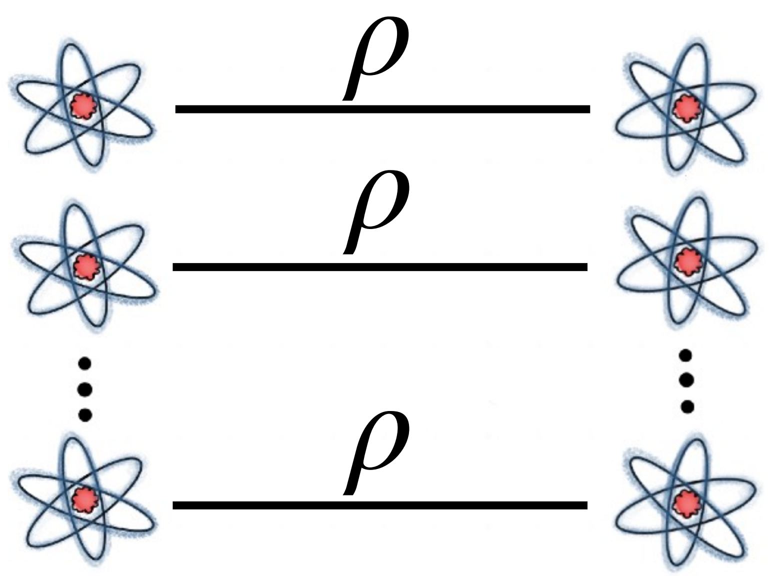


After n uses:



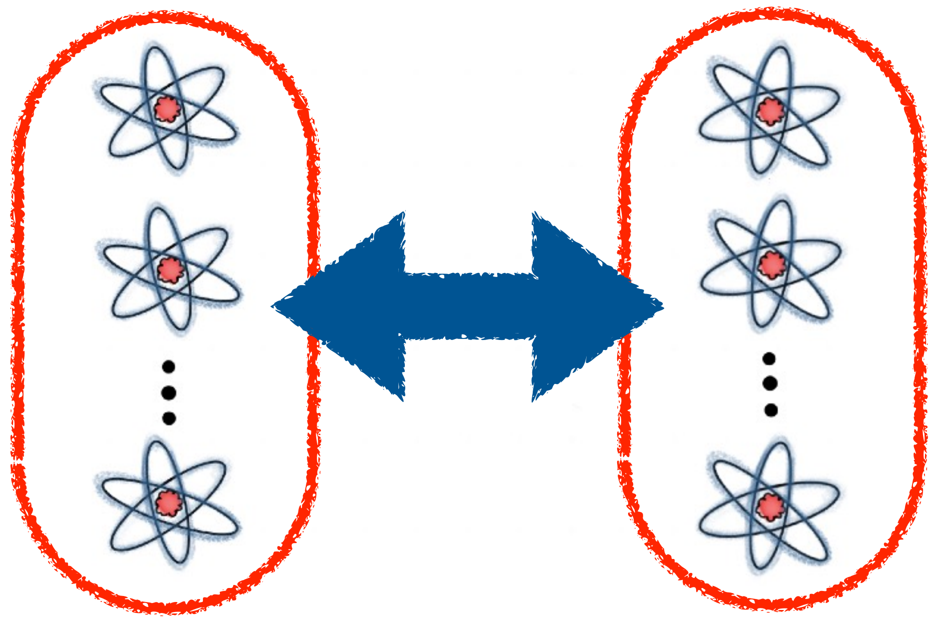


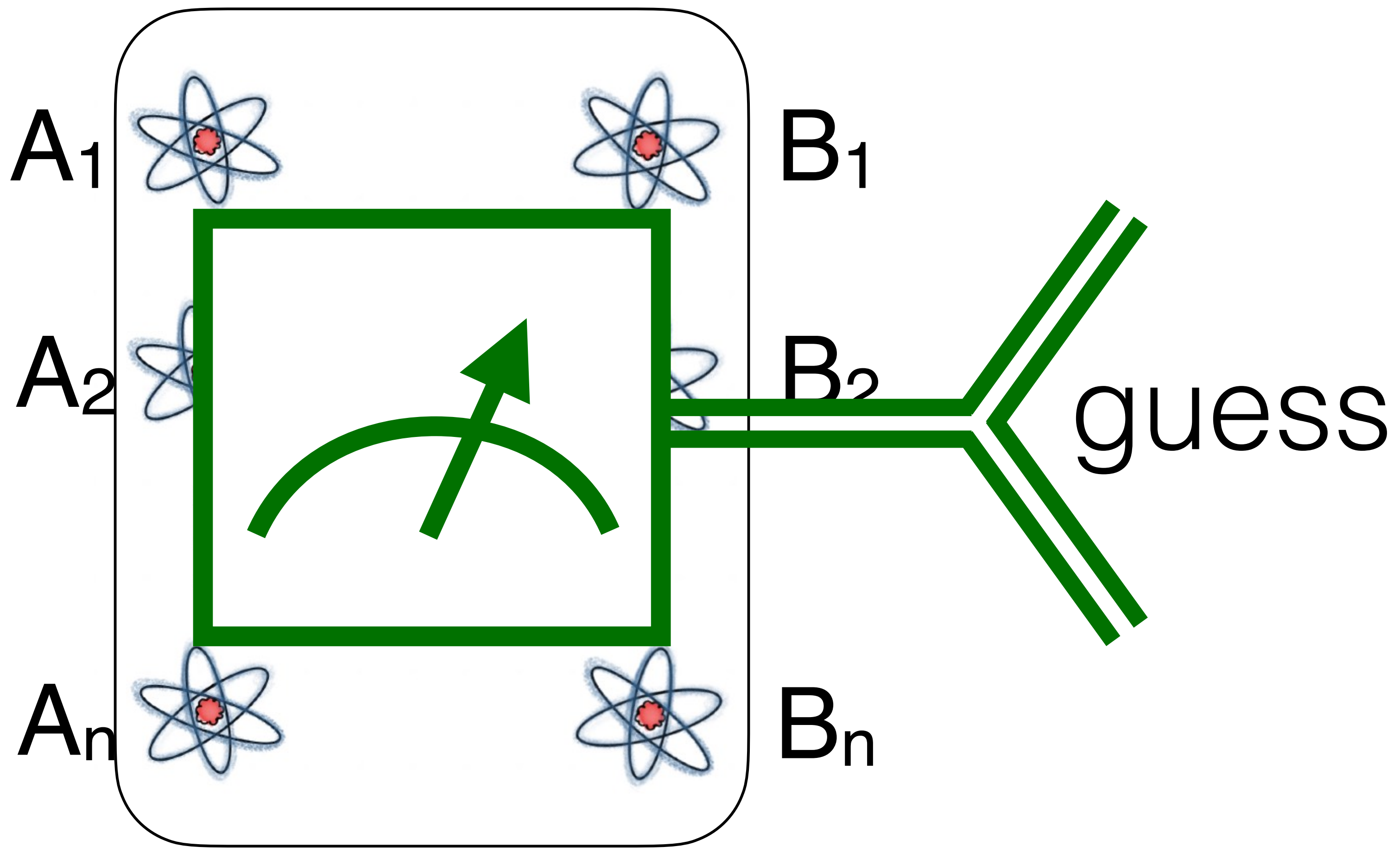
Hypothesis 1: $\rho_{AB}^{\otimes n}$



vs

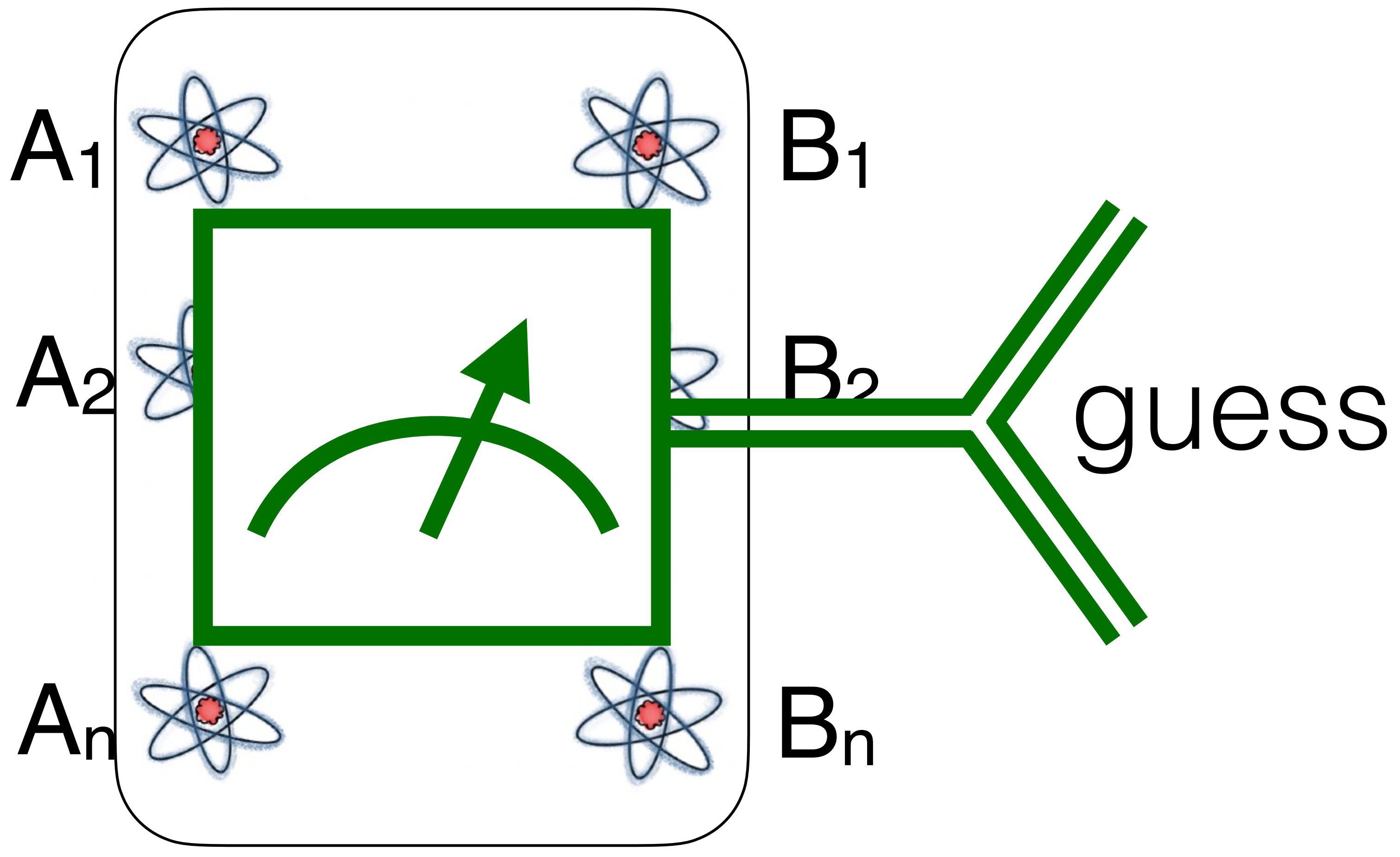
Hypothesis 2: $\sigma_{A^nB^n} \in \mathcal{S}_{A^n:B^n}$





Type 1 error: was $\rho_{AB}^{\otimes n}$, guessed separable

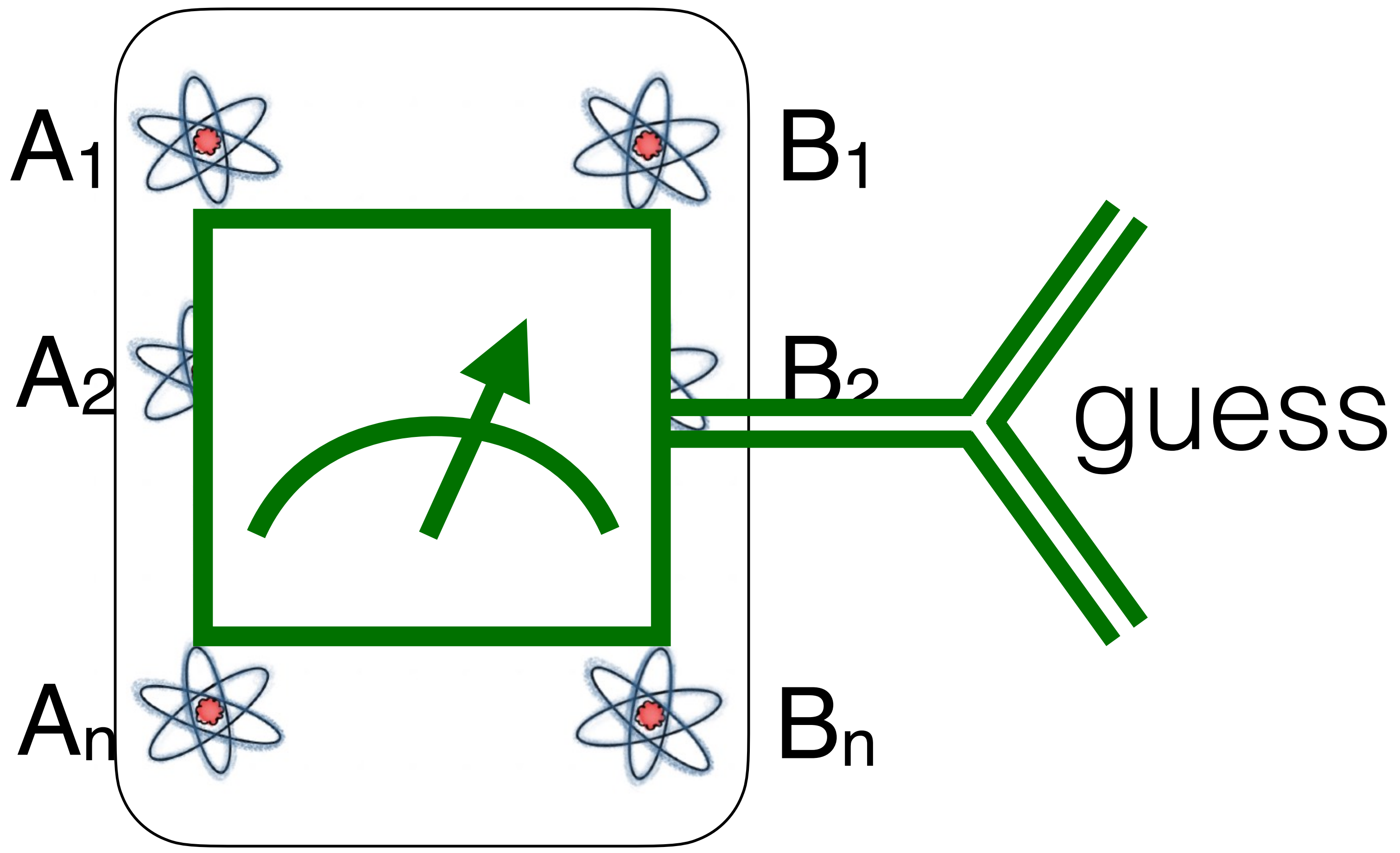
Type 2 error: was separable, guessed $\rho_{AB}^{\otimes n}$



Type 1 error: was $\rho_{AB}^{\otimes n}$, guessed separable

Type 2 error: was separable, guessed $\rho_{AB}^{\otimes n}$

Errors are **not** equally consequential! Minimise Pr{type 2} with Pr{type 1} ≤ ε:

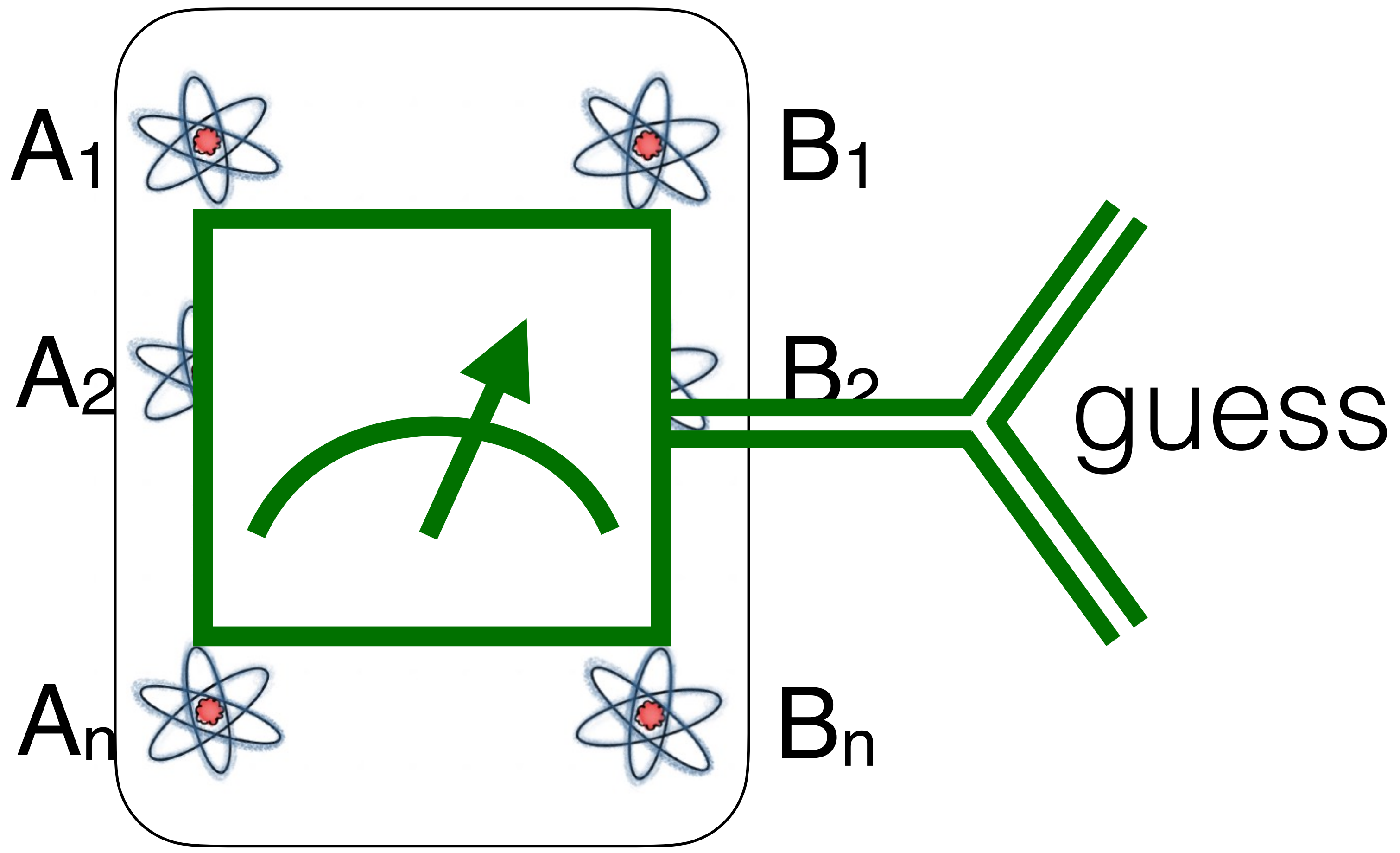


Type 1 error: was $\rho_{AB}^{\otimes n}$, guessed separable

Type 2 error: was separable, guessed $\rho_{AB}^{\otimes n}$

Errors are **not** equally consequential! Minimise Pr{type 2} with Pr{type 1} ≤ ε:

$$\beta_\varepsilon(\rho\|\mathcal{S}) := \min\left\{\max_{\sigma\in\mathcal{S}} \operatorname{Tr}\sigma M :\ \ 0 \leq M \leq I,\ \ \operatorname{Tr}\rho M \geq 1-\varepsilon\right\} \qquad (n=1)$$



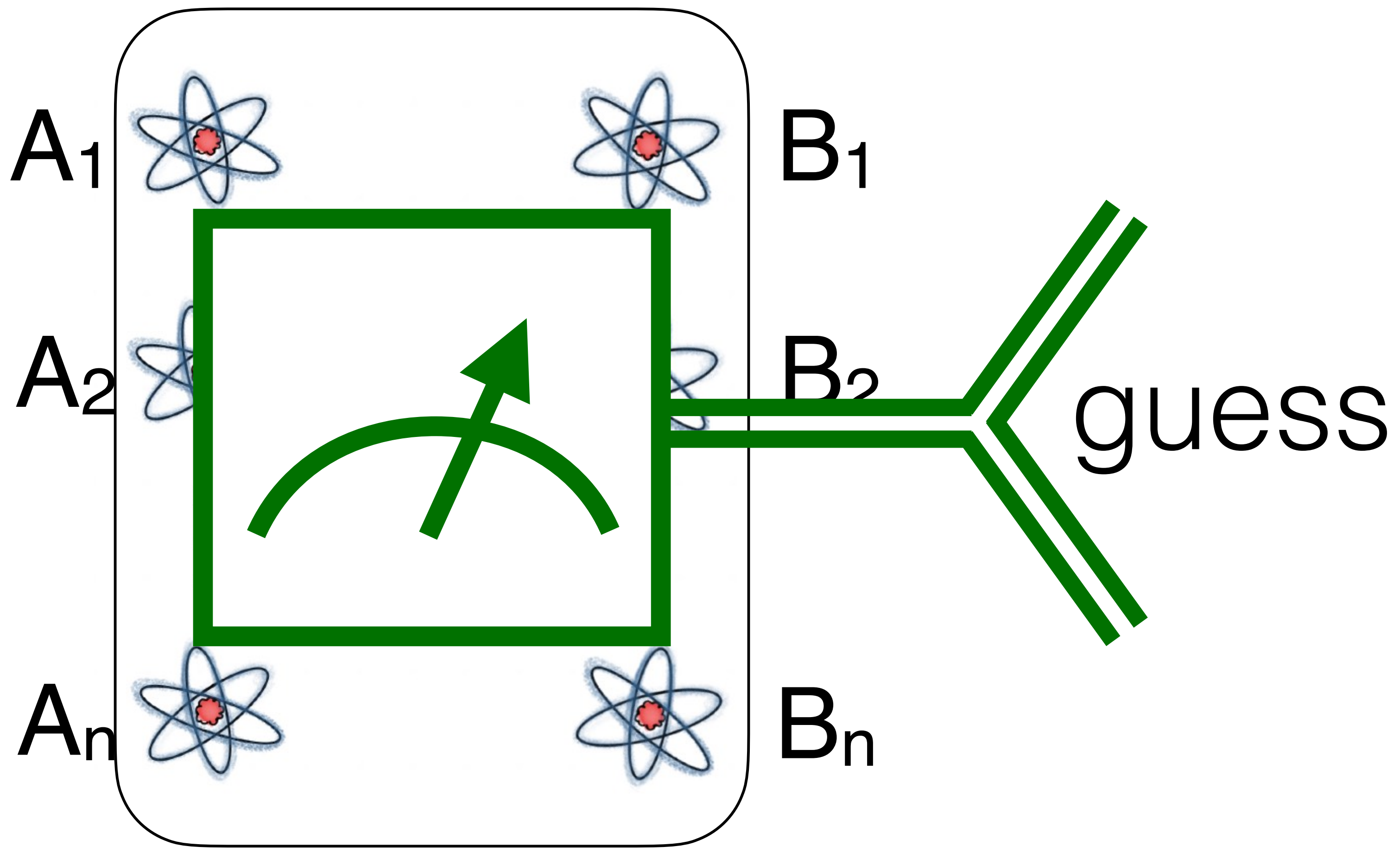
Type 1 error: was $\rho_{AB}^{\otimes n}$, guessed separable

Type 2 error: was separable, guessed $\rho_{AB}^{\otimes n}$

Errors are **not** equally consequential! Minimise Pr{type 2} with Pr{type 1} ≤ ε:

$$\beta_\varepsilon(\rho\|\mathcal{S}) := \min\left\{\max_{\sigma\in\mathcal{S}} \operatorname{Tr}\sigma M : \ 0 \leq M \leq I,\ \ \operatorname{Tr}\rho M \geq 1-\varepsilon\right\} \qquad (n=1)$$

What to expect: $\beta_\varepsilon(\rho^{\otimes n}\,\|\,\mathcal{S}_n) \sim 2^{-cn}$. Find optimal c, called **Stein exponent**:



Type 1 error: was $\rho_{AB}^{\otimes n}$, guessed separable

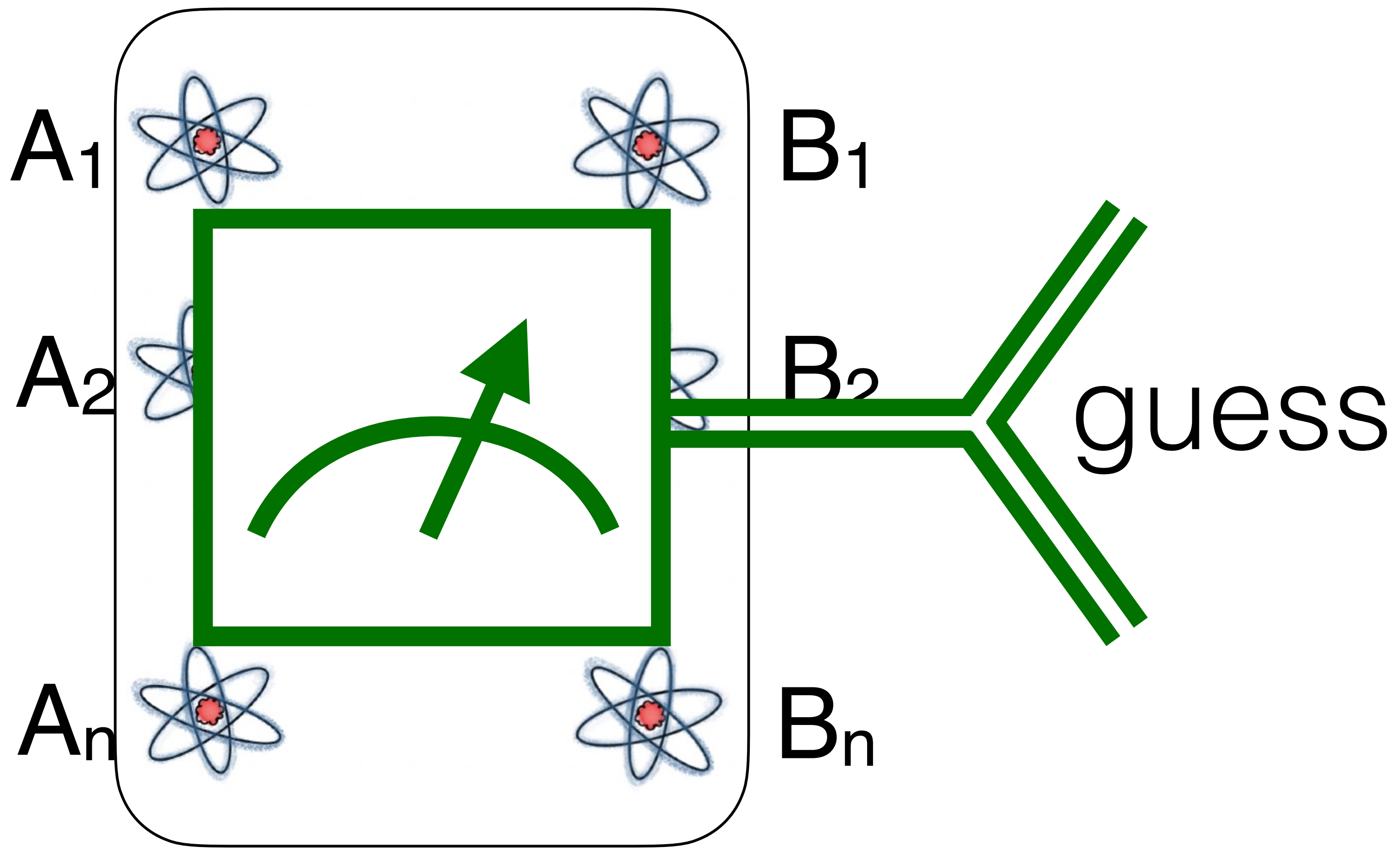
Type 2 error: was separable, guessed $\rho_{AB}^{\otimes n}$

Errors are **not** equally consequential! Minimise Pr{type 2} with Pr{type 1} ≤ ε:

$$\beta_\varepsilon(\rho\|\mathcal{S}) := \min\left\{\max_{\sigma\in\mathcal{S}} \operatorname{Tr}\sigma M:\ \ 0\leq M\leq I,\ \ \operatorname{Tr}\rho M\geq 1-\varepsilon\right\} \qquad (n=1)$$

What to expect: $\beta_\varepsilon(\rho^{\otimes n}\,\|\,\mathcal{S}_n)\sim 2^{-cn}$. Find optimal c, called **Stein exponent**:

$$\text{Stein}(\rho\|\mathcal{S}) := \lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}D_H^\varepsilon(\rho^{\otimes n}\,\|\,\mathcal{S}_n)\,, \qquad D_H^\varepsilon := -\log\beta_\varepsilon$$



Type 1 error: was $\rho_{AB}^{\otimes n}$, guessed separable

Type 2 error: was separable, guessed $\rho_{AB}^{\otimes n}$

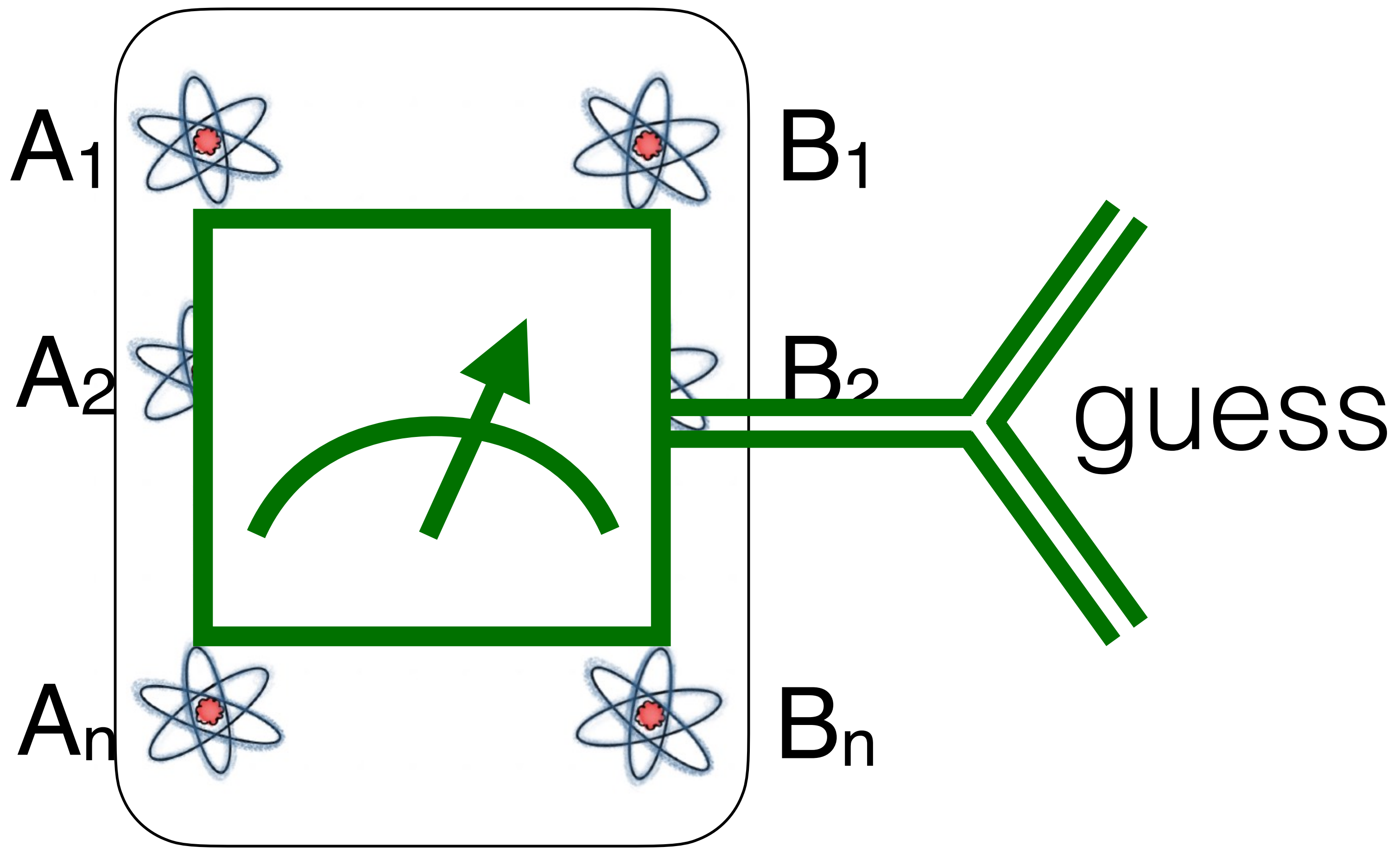
Errors are **not** equally consequential! Minimise Pr{type 2} with Pr{type 1} ≤ ε:

$$\beta_\varepsilon(\rho\|\mathcal{S}) := \min\left\{\max_{\sigma\in\mathcal{S}} \operatorname{Tr}\sigma M : \ 0 \le M \le I, \ \operatorname{Tr}\rho M \ge 1-\varepsilon\right\} \qquad (n=1)$$

What to expect: $\beta_\varepsilon(\rho^{\otimes n}\,\|\,\mathcal{S}_n) \sim 2^{-cn}$. Find optimal *c*, called **Stein exponent**:

$$\operatorname{Stein}(\rho\|\mathcal{S}) := \lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}D_H^\varepsilon(\rho^{\otimes n}\,\|\,\mathcal{S}_n)\,, \qquad D_H^\varepsilon := -\log\beta_\varepsilon$$

Quantifies "ultimate" performance of ent. testing



Type 1 error: was $\rho_{AB}^{\otimes n}$, guessed separable

Type 2 error: was separable, guessed $\rho_{AB}^{\otimes n}$

Errors are **not** equally consequential! Minimise Pr{type 2} with Pr{type 1} ≤ ε:

$$\beta_\varepsilon(\rho\|\mathcal{S}) := \min \left\{ \max_{\sigma\in\mathcal{S}} \operatorname{Tr}\sigma M : \ 0 \leq M \leq I, \ \operatorname{Tr}\rho M \geq 1-\varepsilon \right\} \qquad (n=1)$$

What to expect: $\beta_\varepsilon(\rho^{\otimes n} \,\|\, \mathcal{S}_n) \sim 2^{-cn}$. Find optimal c, called **Stein exponent**:

$$\mathrm{Stein}(\rho\|\mathcal{S}) := \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} D_H^\varepsilon(\rho^{\otimes n} \,\|\, \mathcal{S}_n), \qquad D_H^\varepsilon := -\log \beta_\varepsilon$$

Quantifies "ultimate" performance of ent. testing ⟶ How to calculate it?

Relative entropy of entanglement

Relative entropy: Stein exponent between two **fixed** quantum states.

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\left(\log\rho - \log\sigma\right)\right]$$

Relative entropy of entanglement

Relative entropy: Stein exponent between two **fixed** quantum states.

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\left(\log\rho - \log\sigma\right)\right]$$

Minimise over separable states ⇝ *relative entropy of entanglement.*

$$D(\rho\|\mathcal{S}) := \min_{\sigma_{AB}\in\mathcal{S}_{A:B}} D(\rho_{AB}\,\|\,\sigma_{AB})$$

Vedral, Plenio, Rippin, Knight, PRL 1997

Relative entropy of entanglement

Relative entropy: Stein exponent between two **fixed** quantum states.

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\left(\log\rho - \log\sigma\right)\right]$$

Minimise over separable states $\rightsquigarrow$ *relative entropy of entanglement.*

$$D(\rho\|\mathcal{S}) := \min_{\sigma_{AB}\in\mathcal{S}_{A:B}} D(\rho_{AB}\,\|\,\sigma_{AB})$$

Asymptotic limit:

Vedral, Plenio, Rippin, Knight, PRL 1997

Relative entropy of entanglement

Relative entropy: Stein exponent between two **fixed** quantum states.

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\,(\log\rho - \log\sigma)\right]$$

Minimise over separable states $\rightsquigarrow$ *relative entropy of entanglement.*

$$D(\rho\|\mathcal{S}) := \min_{\sigma_{AB}\in\mathcal{S}_{A:B}} D(\rho_{AB}\,\|\,\sigma_{AB})$$

Asymptotic limit:

⇕

Thermodynamic limit

Vedral, Plenio, Rippin, Knight, PRL 1997

Relative entropy of entanglement

Relative entropy: Stein exponent between two **fixed** quantum states.

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\left(\log\rho - \log\sigma\right)\right]$$

Minimise over separable states ⇝ *relative entropy of entanglement.*

$$D(\rho\|\mathcal{S}) := \min_{\sigma_{AB}\in\mathcal{S}_{A:B}} D(\rho_{AB}\,\|\,\sigma_{AB})$$

regularisation

Asymptotic limit:

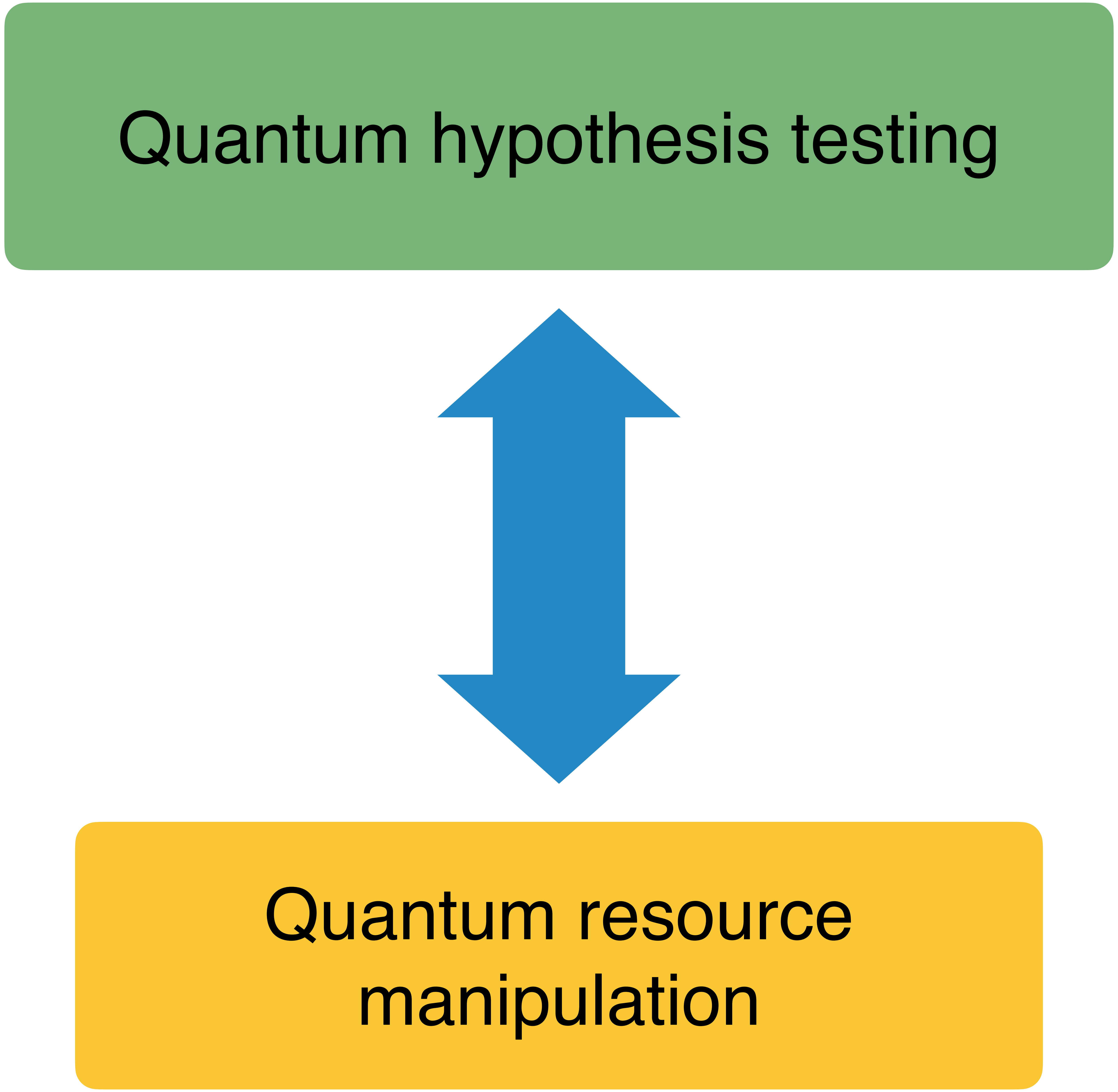
$$D^{\infty}(\rho\|\mathcal{S}) := \lim_{n\to\infty}\frac{1}{n}D\left(\rho_{AB}^{\otimes n}\,\|\,\mathcal{S}_{A^n:B^n}\right)$$

↕ Thermodynamic limit

Vedral, Plenio, Rippin, Knight, PRL 1997

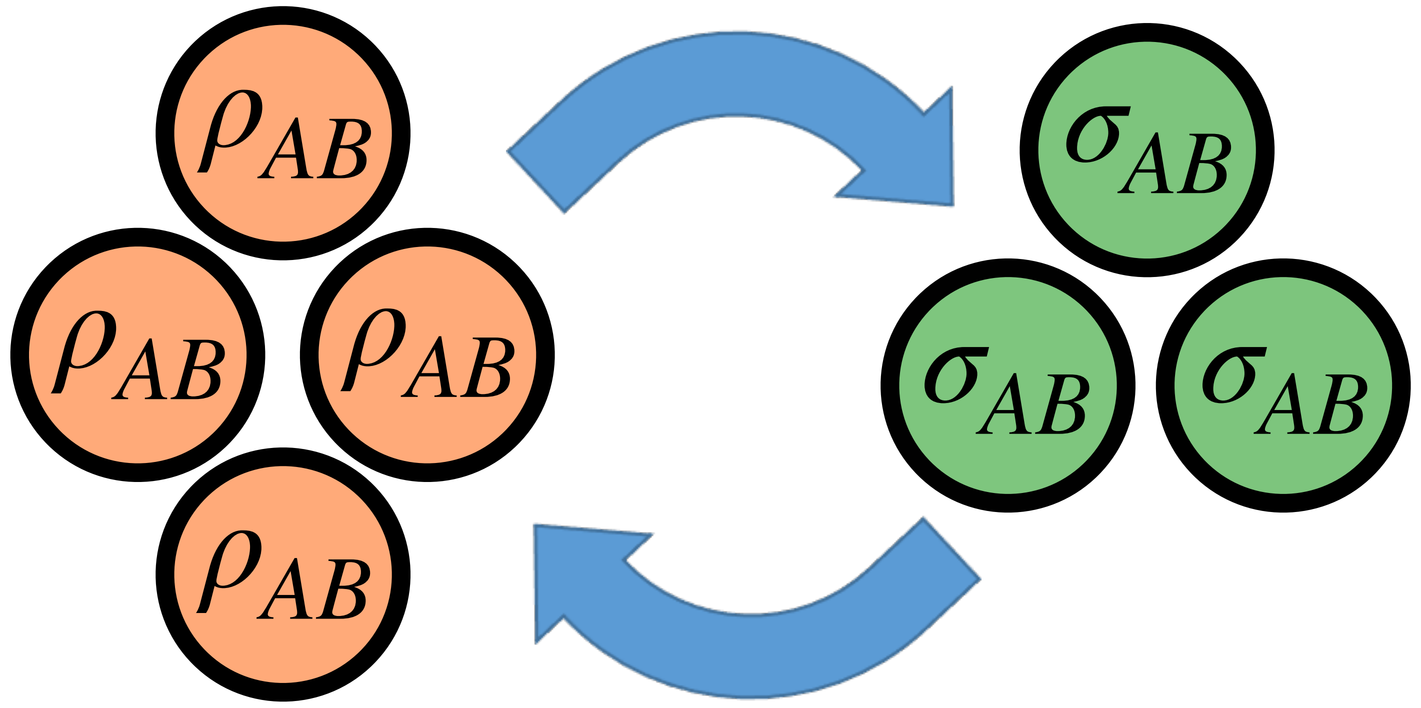
Quantum hypothesis testing

Quantum resource manipulation


Quantum hypothesis testing
Quantum resource manipulation

The quest for reversibility

Plenio's problem: Does there exist a "thermodynamical" theory of entanglement?

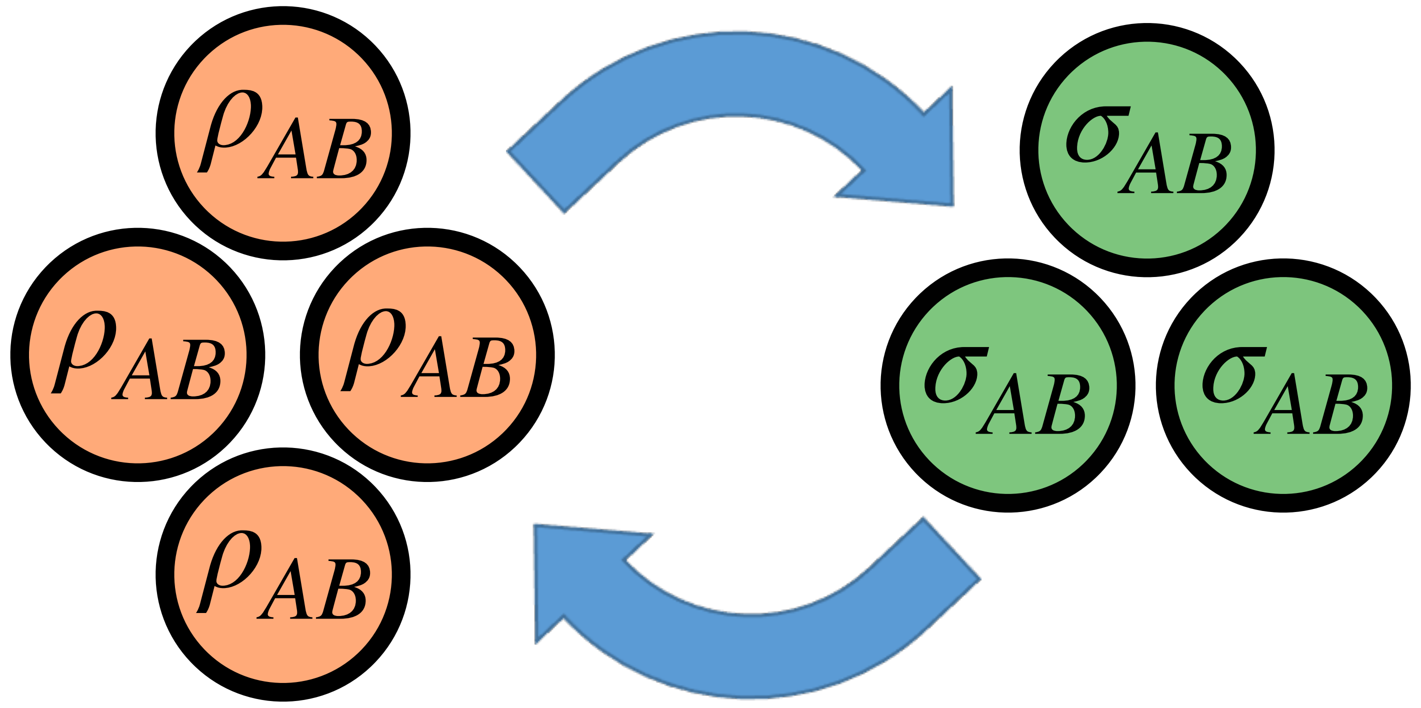


Brandão and Plenio, Nature Physics 2008 & CMP 2010.

L and Regula, Nature Physics 2023

The quest for reversibility

Plenio's problem: Does there exist a "thermodynamical" theory of entanglement?



Reformulation: does there exist a set of operations $\mathcal{O}$ such that

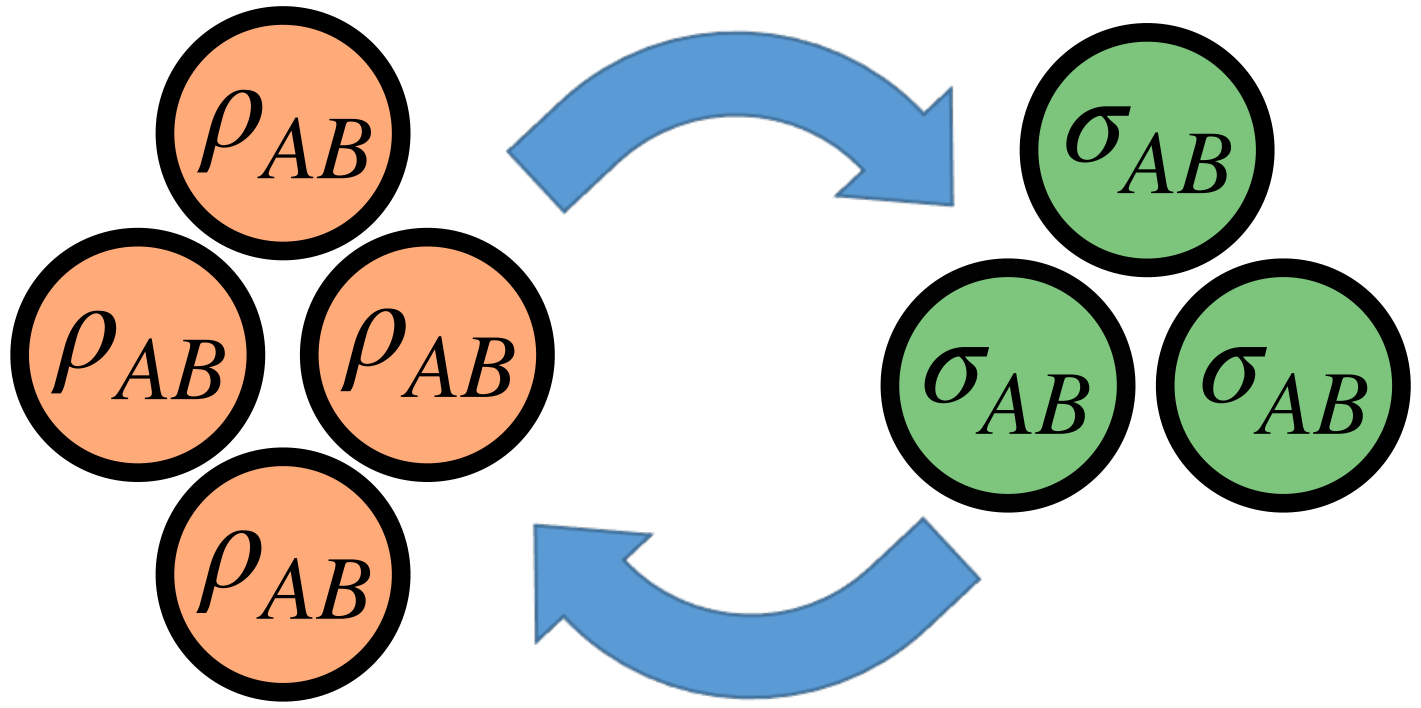
$$R_{\mathcal{O}}(\rho_{AB} \to \sigma_{AB}) = \frac{E(\rho_{AB})}{E(\sigma_{AB})}$$

Brandão and Plenio, Nature Physics 2008 & CMP 2010.

L and Regula, Nature Physics 2023

The quest for reversibility

Plenio's problem: Does there exist a "thermodynamical" theory of entanglement?



Reformulation: does there exist a set of operations $\mathscr{O}$ such that

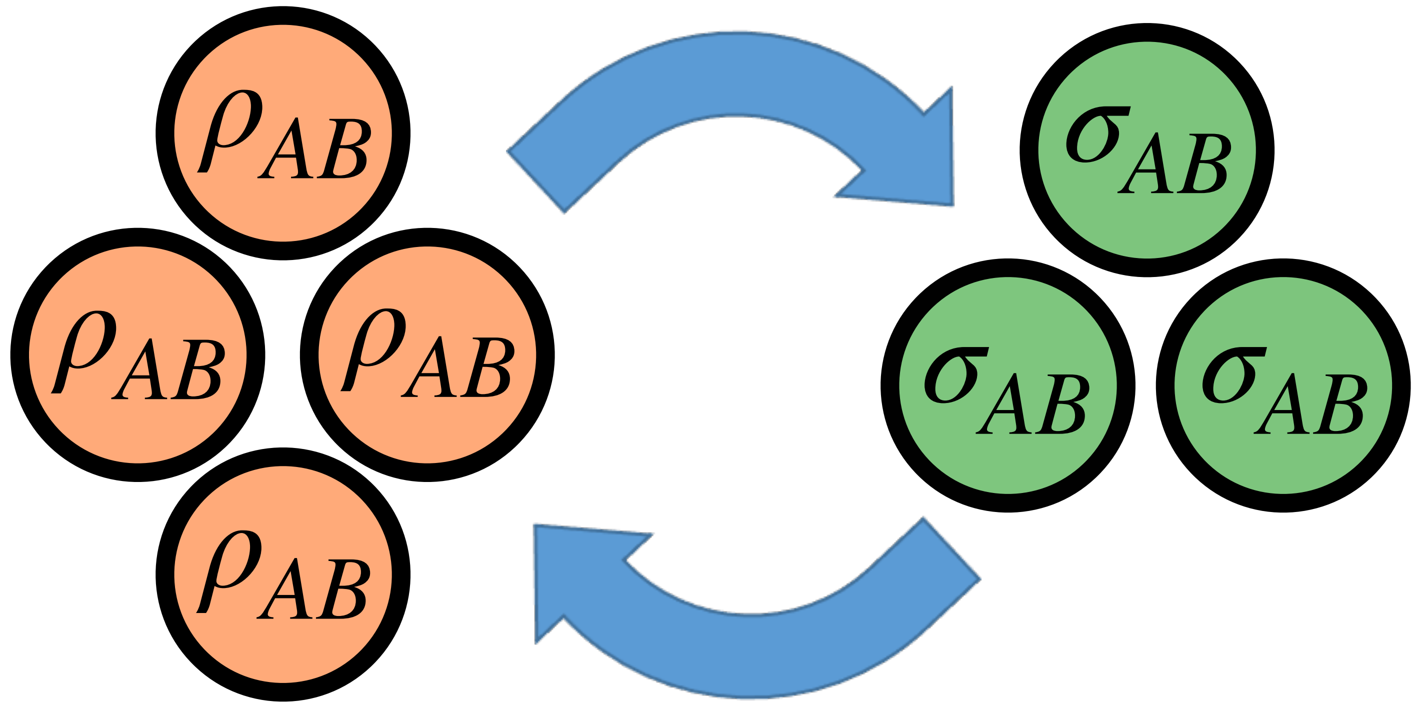
$$R_{\mathscr{O}}(\rho_{AB} \to \sigma_{AB}) = \frac{E(\rho_{AB})}{E(\sigma_{AB})} = R_{\mathscr{O}}(\sigma_{AB} \to \rho_{AB})^{-1}$$

Brandão and Plenio, Nature Physics 2008 & CMP 2010.

L and Regula, Nature Physics 2023

The quest for reversibility

Plenio's problem: Does there exist a "thermodynamical" theory of entanglement?



Reformulation: does there exist a set of operations $\mathcal{O}$ such that

$$R_{\mathcal{O}}(\rho_{AB} \to \sigma_{AB}) = \frac{E(\rho_{AB})}{E(\sigma_{AB})} = R_{\mathcal{O}}(\sigma_{AB} \to \rho_{AB})^{-1}$$

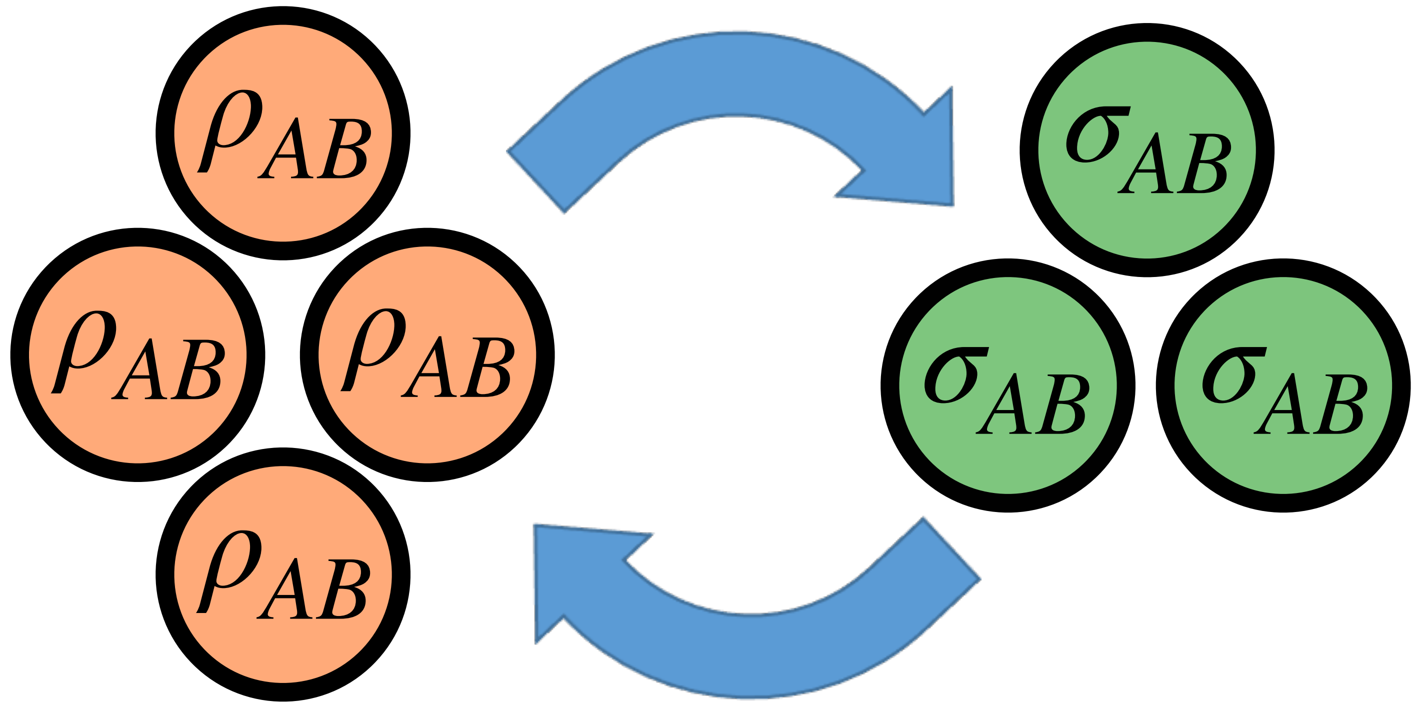
True for pure states with $E(\psi_{AB}) = S(\psi_A)$, but false for mixed states under LOCCs.

Bennett, Bernstein, Popescu, and Schumacher, PRA 1996.
Brandão and Plenio, Nature Physics 2008 & CMP 2010.

Vidal and Cirac, PRL 2001
L and Regula, Nature Physics 2023

The quest for reversibility

Plenio's problem: Does there exist a "thermodynamical" theory of entanglement?



Reformulation: does there exist a set of operations $\mathcal{O}$ such that

One entanglement measure to rule them all!

$$R_{\mathcal{O}}(\rho_{AB} \to \sigma_{AB}) = \frac{E(\rho_{AB})}{E(\sigma_{AB})} = R_{\mathcal{O}}(\sigma_{AB} \to \rho_{AB})^{-1}$$

True for pure states with $E(\psi_{AB}) = S(\psi_A)$, but false for mixed states under LOCCs.

Bennett, Bernstein, Popescu, and Schumacher, PRA 1996.

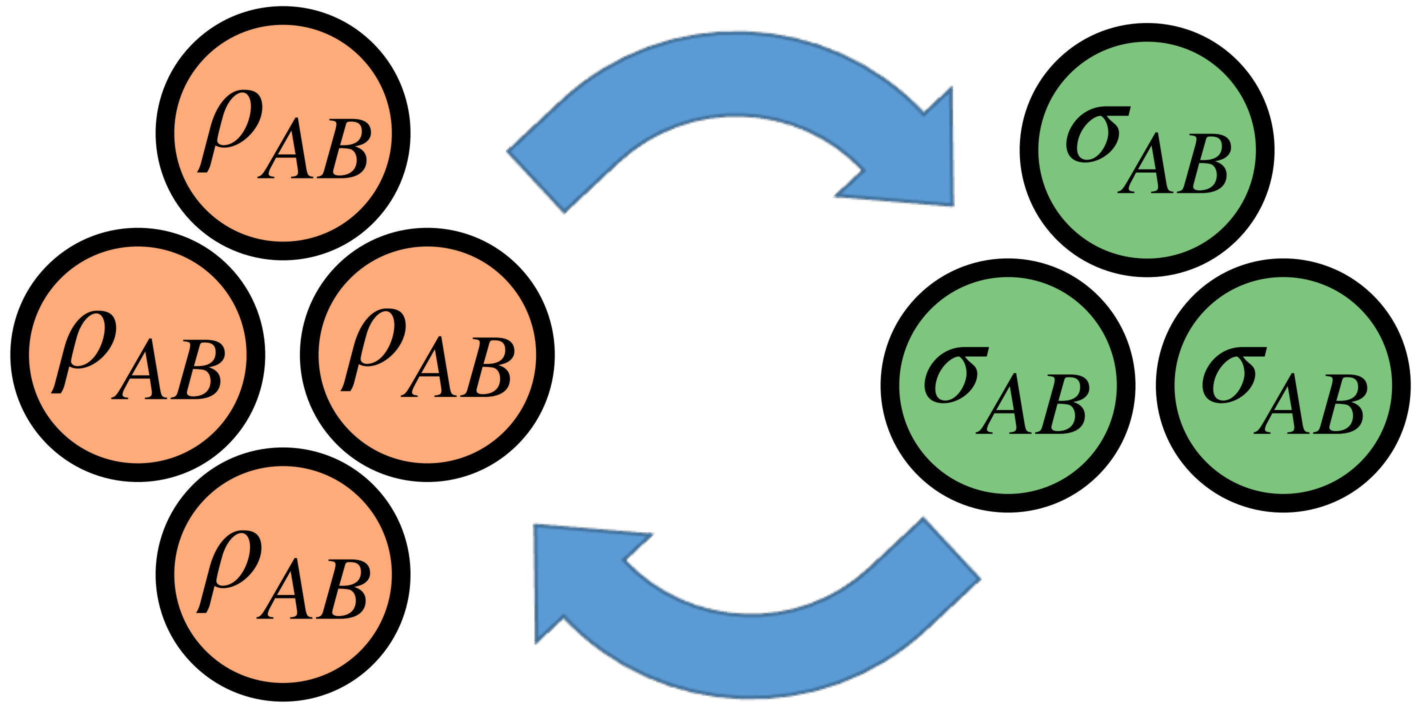
Brandão and Plenio, Nature Physics 2008 & CMP 2010.

Vidal and Cirac, PRL 2001

L and Regula, Nature Physics 2023

The quest for reversibility

Plenio's problem: Does there exist a "thermodynamical" theory of entanglement?



Reformulation: does there exist a set of operations $\mathcal{O}$ such that

One entanglement measure to rule them all!

$$R_{\mathcal{O}}(\rho_{AB} \to \sigma_{AB}) = \frac{E(\rho_{AB})}{E(\sigma_{AB})} = R_{\mathcal{O}}(\sigma_{AB} \to \rho_{AB})^{-1}$$

Entropy in thermodynamics!

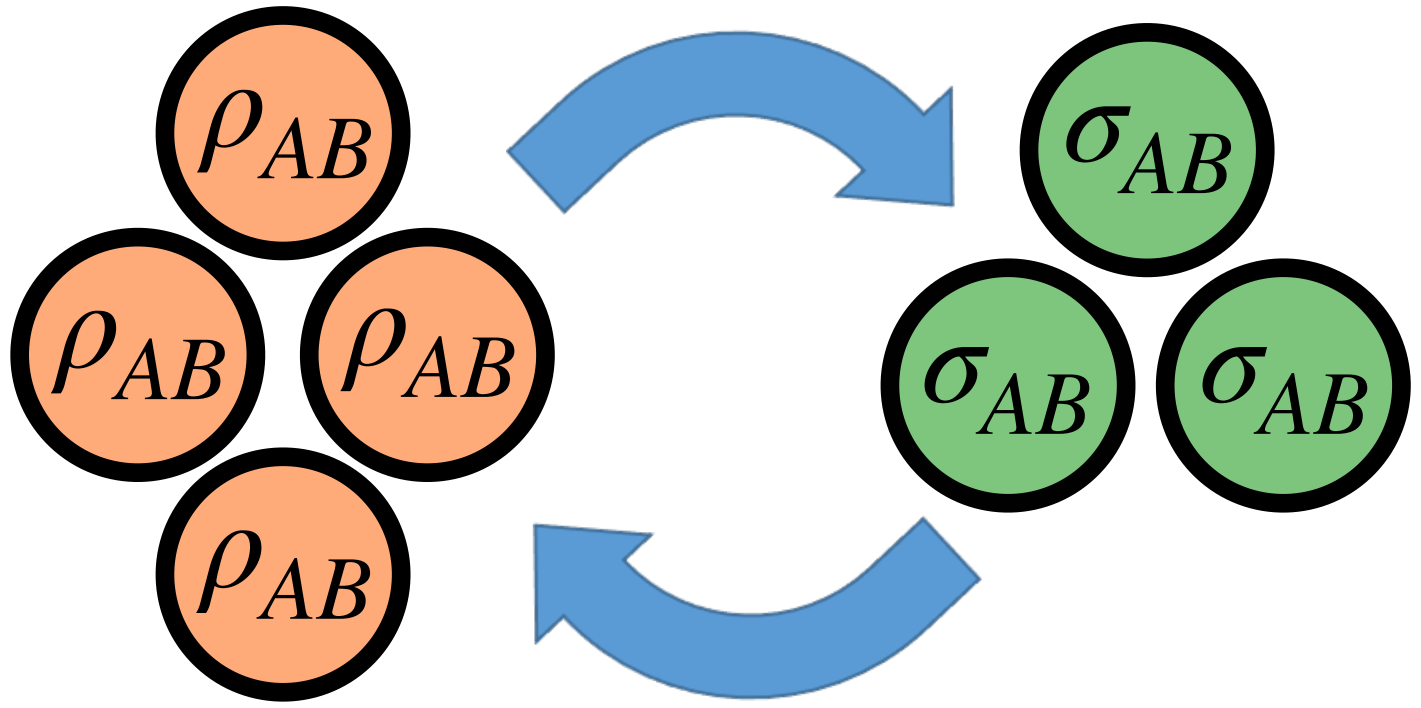
True for pure states with $E(\psi_{AB}) = S(\psi_A)$, but false for mixed states under LOCCs.

Bennett, Bernstein, Popescu, and Schumacher, PRA 1996.
Brandão and Plenio, Nature Physics 2008 & CMP 2010.

Vidal and Cirac, PRL 2001
L and Regula, Nature Physics 2023

The quest for reversibility

Plenio's problem: Does there exist a "thermodynamical" theory of entanglement?



Reformulation: does there exist a set of operations $\mathcal{O}$ such that

One entanglement measure to rule them all!

$$R_{\mathcal{O}}(\rho_{AB} \to \sigma_{AB}) = \frac{E(\rho_{AB})}{E(\sigma_{AB})} = R_{\mathcal{O}}(\sigma_{AB} \to \rho_{AB})^{-1}$$

→ "Second law" of q. resource theories

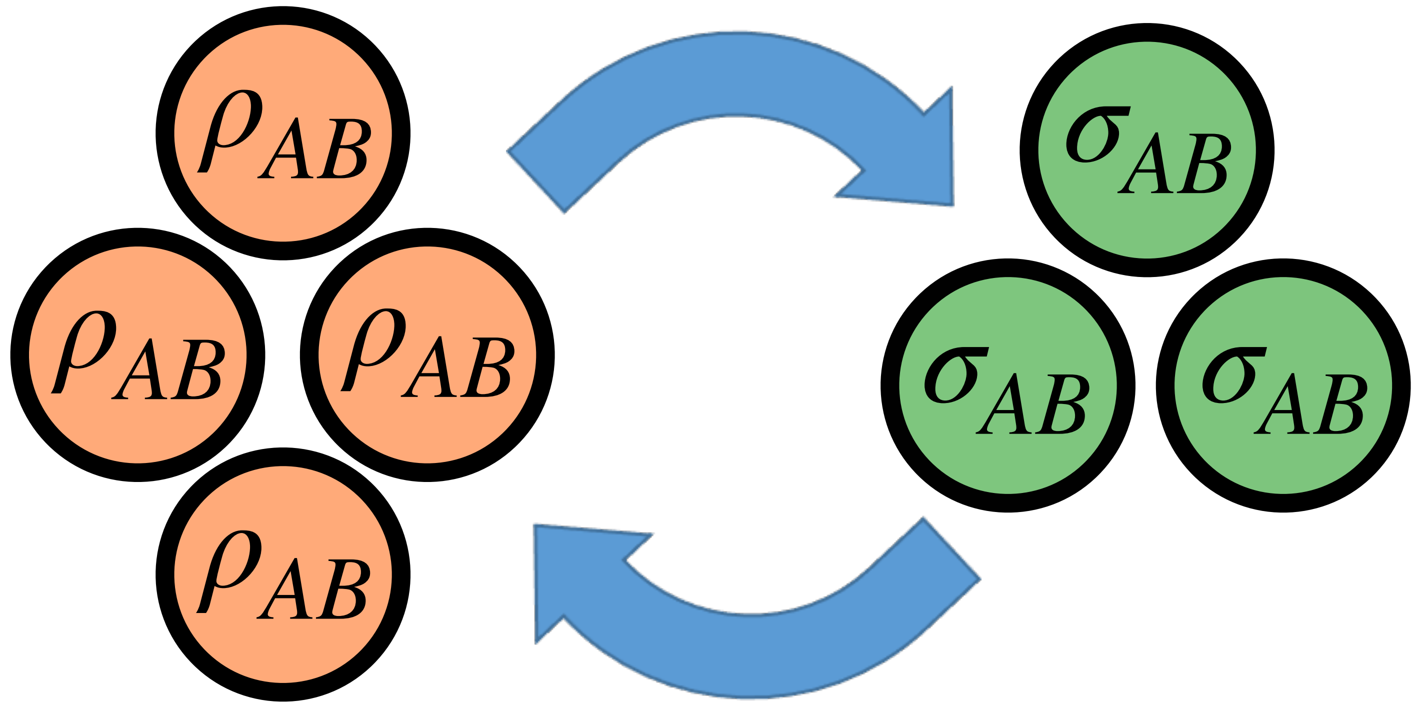
True for pure states with $E(\psi_{AB}) = S(\psi_A)$, but false for mixed states under LOCCs.

Bennett, Bernstein, Popescu, and Schumacher, PRA 1996.
Brandão and Plenio, Nature Physics 2008 & CMP 2010.

Vidal and Cirac, PRL 2001
L and Regula, Nature Physics 2023

The quest for reversibility

Plenio's problem: Does there exist a "thermodynamical" theory of entanglement?

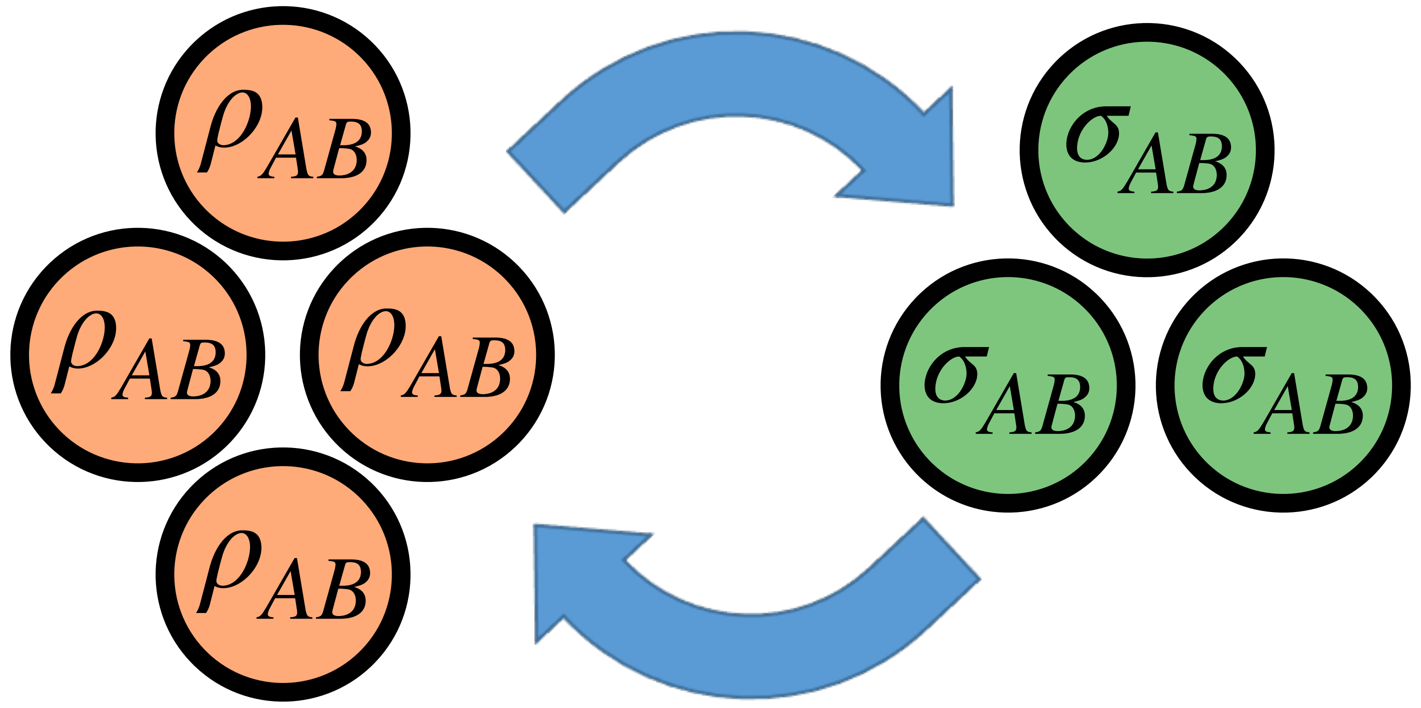


Brandão and Plenio, Nature Physics 2008 & CMP 2010.

L and Regula, Nature Physics 2023

The quest for reversibility

Plenio's problem: Does there exist a "thermodynamical" theory of entanglement?



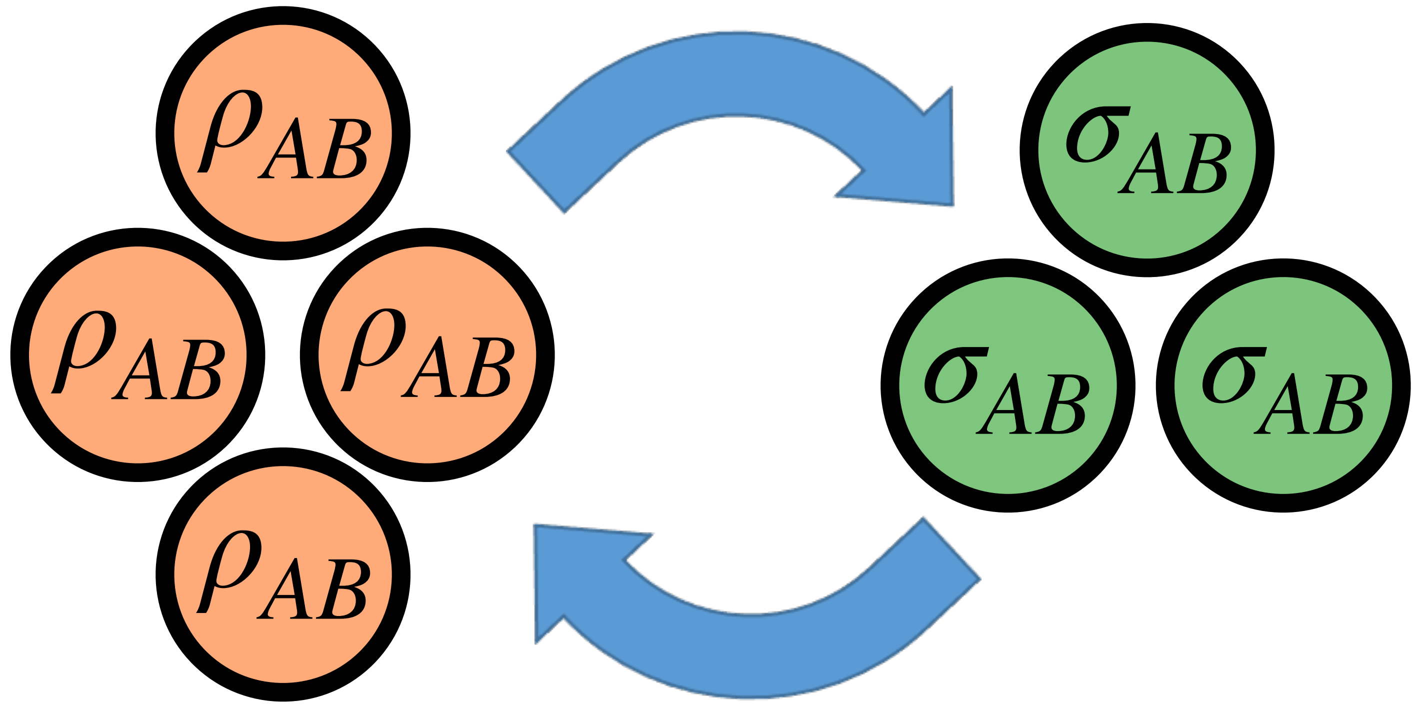
Brandão and Plenio: $E_{c,\,\mathrm{ANE}}(\rho_{AB}) = D^{\infty}(\rho\|\mathcal{S}),\quad E_{d,\,\mathrm{ANE}}(\rho_{AB}) = \mathrm{Stein}(\rho\|\mathcal{S})$

Brandão and Plenio, Nature Physics 2008 & CMP 2010.

L and Regula, Nature Physics 2023

The quest for reversibility

Plenio's problem: Does there exist a "thermodynamical" theory of entanglement?



Brandão and Plenio: $E_{c,\,\mathrm{ANE}}(\rho_{AB}) = D^\infty(\rho\|\mathcal{S}), \quad E_{d,\,\mathrm{ANE}}(\rho_{AB}) = \mathrm{Stein}(\rho\|\mathcal{S})$

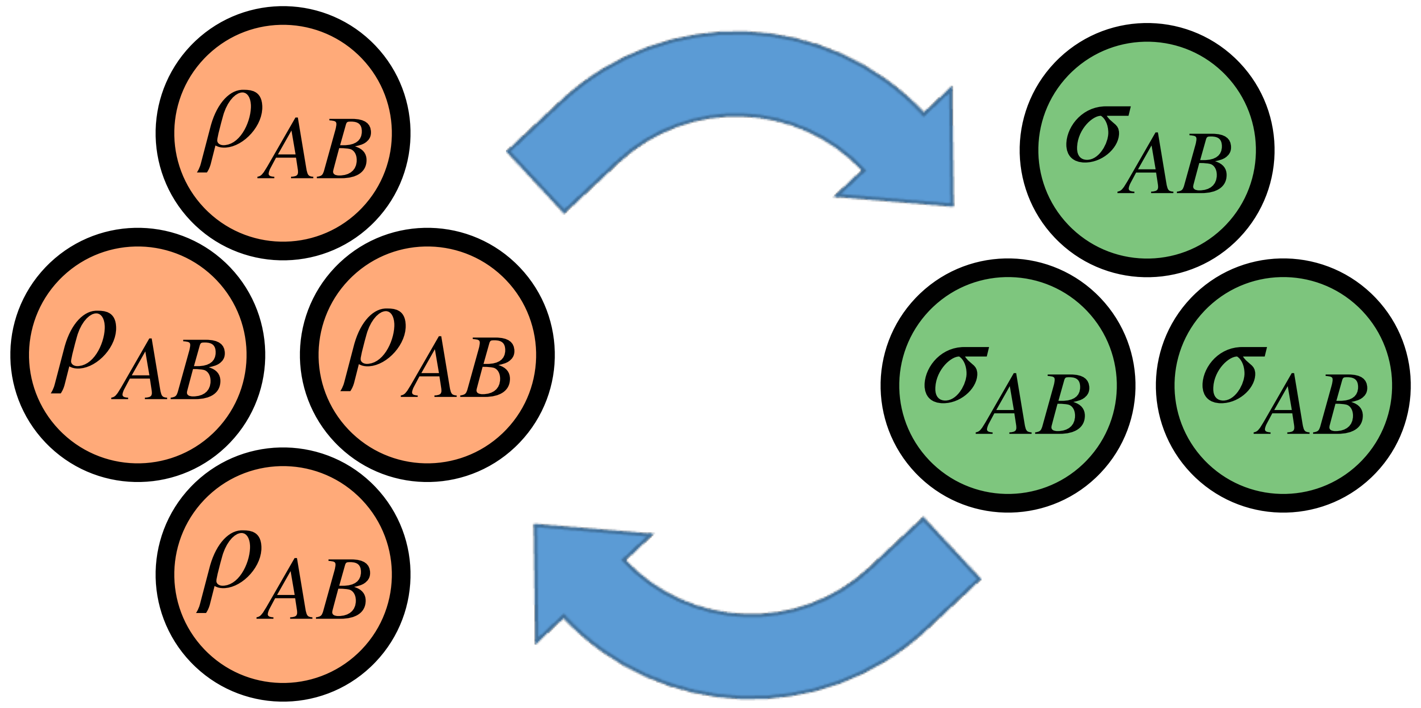
If $\mathrm{Stein}(\rho\|\mathcal{S}) \overset{?}{=} D^\infty(\rho\|\mathcal{S}) \implies E_{d,\,\mathrm{ANE}}(\rho) = E_{c,\,\mathrm{ANE}}(\rho) = D^\infty(\rho\|\mathcal{S})$.

Brandão and Plenio, Nature Physics 2008 & CMP 2010.

L and Regula, Nature Physics 2023

The quest for reversibility

Plenio's problem: Does there exist a "thermodynamical" theory of entanglement?



Brandão and Plenio: $E_{c,\,\mathrm{ANE}}(\rho_{AB}) = D^\infty(\rho\|\mathcal{S}),\quad E_{d,\,\mathrm{ANE}}(\rho_{AB}) = \mathrm{Stein}(\rho\|\mathcal{S})$

If $\mathrm{Stein}(\rho\|\mathcal{S}) \overset{?}{=} D^\infty(\rho\|\mathcal{S})$ $\Longrightarrow$ $E_{d,\,\mathrm{ANE}}(\rho) = E_{c,\,\mathrm{ANE}}(\rho) = D^\infty(\rho\|\mathcal{S})$.

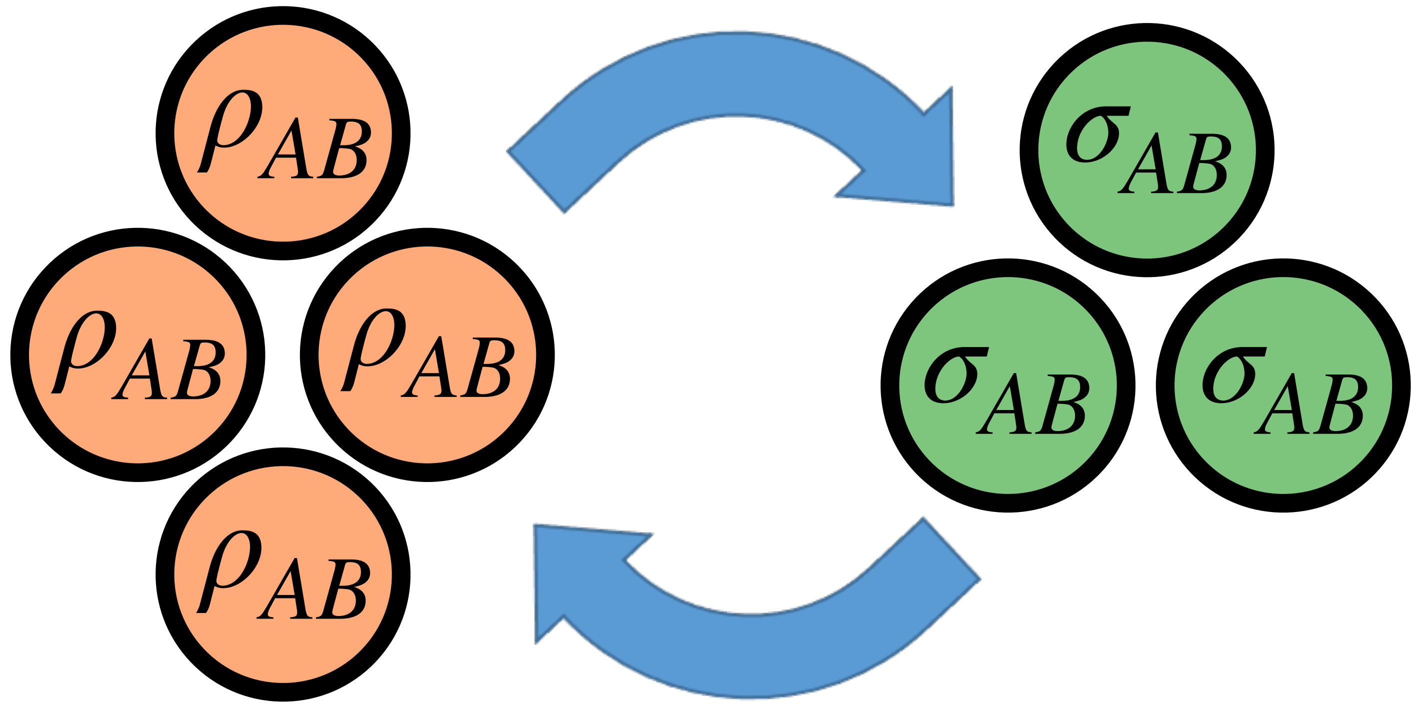
$\Longrightarrow$ Reversibility under ANE!

Brandão and Plenio, Nature Physics 2008 & CMP 2010.

L and Regula, Nature Physics 2023

The quest for reversibility

Plenio's problem: Does there exist a "thermodynamical" theory of entanglement?



Brandão and Plenio: $E_{c,\,\mathrm{ANE}}(\rho_{AB}) = D^\infty(\rho\|\mathcal{S}),\quad E_{d,\,\mathrm{ANE}}(\rho_{AB}) = \mathrm{Stein}(\rho\|\mathcal{S})$

If $\mathrm{Stein}(\rho\|\mathcal{S}) \overset{?}{=} D^\infty(\rho\|\mathcal{S})$ $\implies$ $E_{d,\,\mathrm{ANE}}(\rho) = E_{c,\,\mathrm{ANE}}(\rho) = D^\infty(\rho\|\mathcal{S})$.

Generalised quantum Stein's lemma $\implies$ Reversibility under ANE!

Brandão and Plenio, Nature Physics 2008 & CMP 2010.

L and Regula, Nature Physics 2023

Main result

Generalised quantum Stein's lemma

$$\mathrm{Stein}(\rho\|\mathscr{S}) = D^\infty(\rho\|\mathscr{S})$$

L, *IEEE Trans. Inf. Theory* 2025 (arXiv:2408.06410).

Hayashi/Yamasaki, arXiv:2408.02722

Main result

Generalised quantum Stein's lemma

$$\mathrm{Stein}(\rho\|\mathscr{S}) = D^{\infty}(\rho\|\mathscr{S})$$

Applies to other quantum resources ➡ Asymptotic reversibility for magic

L, *IEEE Trans. Inf. Theory* 2025 (arXiv:2408.06410).

Hayashi/Yamasaki, arXiv:2408.02722

Main result

Generalised quantum Stein's lemma

$$\mathrm{Stein}(\rho\|\mathcal{S}) = D^{\infty}(\rho\|\mathcal{S})$$

Applies to other quantum resources → Asymptotic reversibility for magic

Regularised relative entropy of entanglement $D^{\infty}(\,\cdot\|\mathcal{S})$ 

L, *IEEE Trans. Inf. Theory* 2025 (arXiv:2408.06410).

Hayashi/Yamasaki, arXiv:2408.02722

Main result

Generalised quantum Stein's lemma +

$$\mathrm{Stein}(\rho\|\mathcal{S}) = D^{\infty}(\rho\|\mathcal{S})$$

Applies to other quantum resources → Asymptotic reversibility for magic

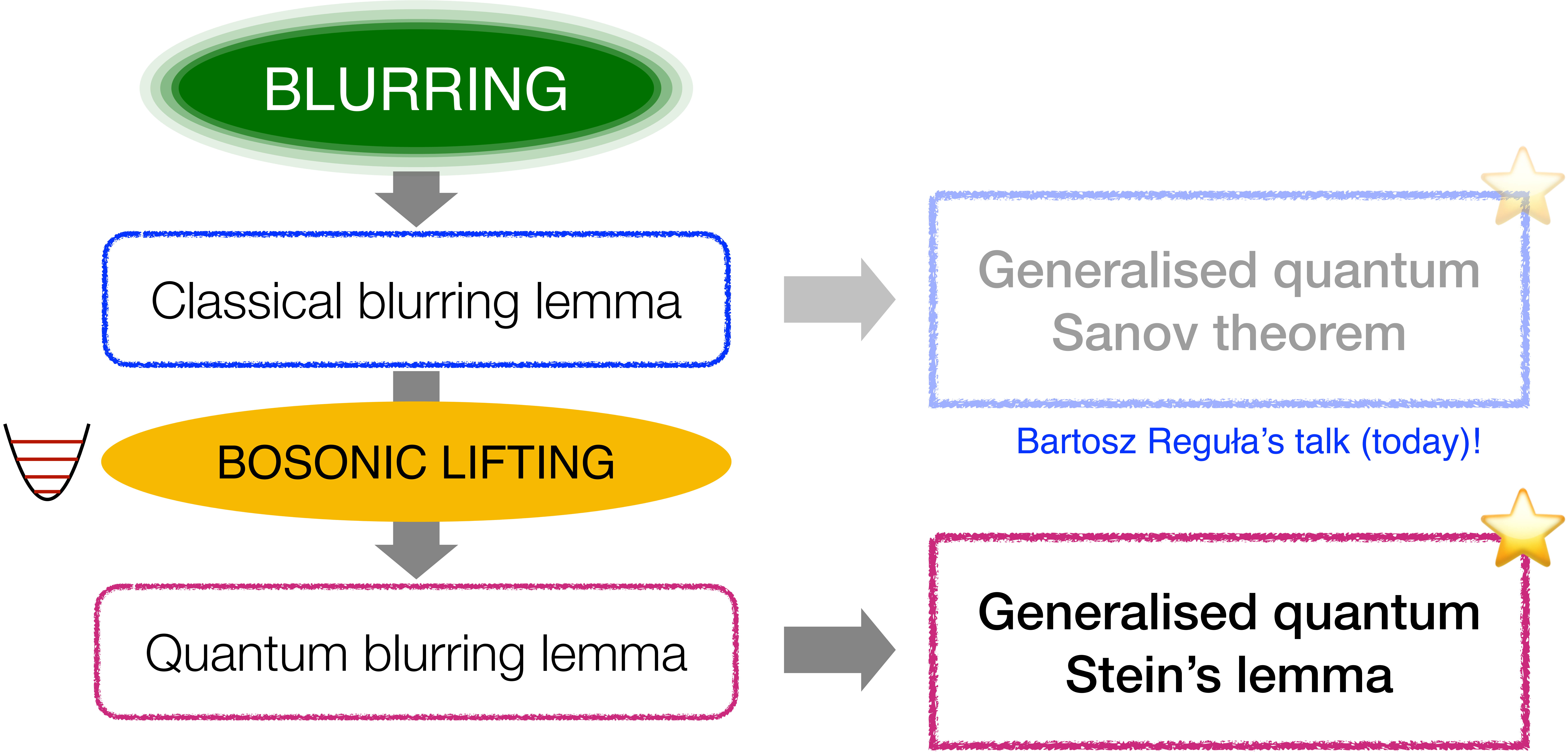
Regularised relative entropy of entanglement $D^{\infty}(\,\cdot\|\mathcal{S})$ 

\+ Solution for almost-i.i.d. source $\rho_n : \rho_n^{(i)} \approx \rho$ apart from const. many sites *i*.

L, *IEEE Trans. Inf. Theory* 2025 (arXiv:2408.06410). Hayashi/Yamasaki, arXiv:2408.02722

Main proof technique: blurring

Proof structure



Step 1: Smoothed max-relative entropy

Smoothed max-relative entropy: $D^{\varepsilon}_{\max}(\rho\|\sigma) := \min_{\rho' \approx_{\varepsilon} \rho} \inf\{\lambda : \rho' \leq 2^{\lambda}\sigma\}$

Brandão/Plenio, CMP 2010; Datta, Int. J. Quantum Inf. 2009; Hayashi/Tomamichel, IEEE 2013.

Step 1: Smoothed max-relative entropy

Smoothed max-relative entropy: $D^{\varepsilon}_{\max}(\rho\|\sigma) := \min_{\rho' \approx_\varepsilon \rho} \inf\{\lambda : \rho' \le 2^\lambda \sigma\}$

trace-norm ball

Brandão/Plenio, CMP 2010; Datta, Int. J. Quantum Inf. 2009; Hayashi/Tomamichel, IEEE 2013.

Step 1: Smoothed max-relative entropy

$2^\lambda \sigma - \rho'$ is positive semi-definite

Smoothed max-relative entropy: $D^\varepsilon_{\max}(\rho\|\sigma) := \min_{\rho' \approx_\varepsilon \rho} \inf\{\lambda : \rho' \leq 2^\lambda \sigma\}$

trace-norm ball

Brandão/Plenio, CMP 2010; Datta, Int. J. Quantum Inf. 2009; Hayashi/Tomamichel, IEEE 2013.

Step 1: Smoothed max-relative entropy

$2^{\lambda}\sigma - \rho'$ is positive semi-definite

Smoothed max-relative entropy: $D^{\varepsilon}_{\max}(\rho\|\sigma) := \min_{\rho' \approx_{\varepsilon} \rho} \inf\{\lambda :\ \rho' \leq 2^{\lambda}\sigma\}$

trace-norm ball

Regularisation: $F_{\rho}(\varepsilon) := \liminf_{n\to\infty} \frac{1}{n} \min_{\sigma_n \in \mathcal{S}_n} D^{\varepsilon}_{\max}(\rho^{\otimes n} \| \sigma_n)$

Brandão/Plenio, CMP 2010; Datta, Int. J. Quantum Inf. 2009; Hayashi/Tomamichel, IEEE 2013.

Step 1: Smoothed max-relative entropy

$2^\lambda\sigma - \rho'$ is positive semi-definite

Smoothed max-relative entropy: $D^\varepsilon_{\max}(\rho\|\sigma) := \min_{\rho' \approx_\varepsilon \rho} \inf\{\lambda :\ \rho' \leq 2^\lambda \sigma\}$

trace-norm ball

Regularisation: $F_\rho(\varepsilon) := \liminf_{n\to\infty} \frac{1}{n} \min_{\sigma_n \in \mathcal{S}_n} D^\varepsilon_{\max}(\rho^{\otimes n} \| \sigma_n)$ ⟿ **non-increasing** in ε.

Brandão/Plenio, CMP 2010; Datta, Int. J. Quantum Inf. 2009; Hayashi/Tomamichel, IEEE 2013.

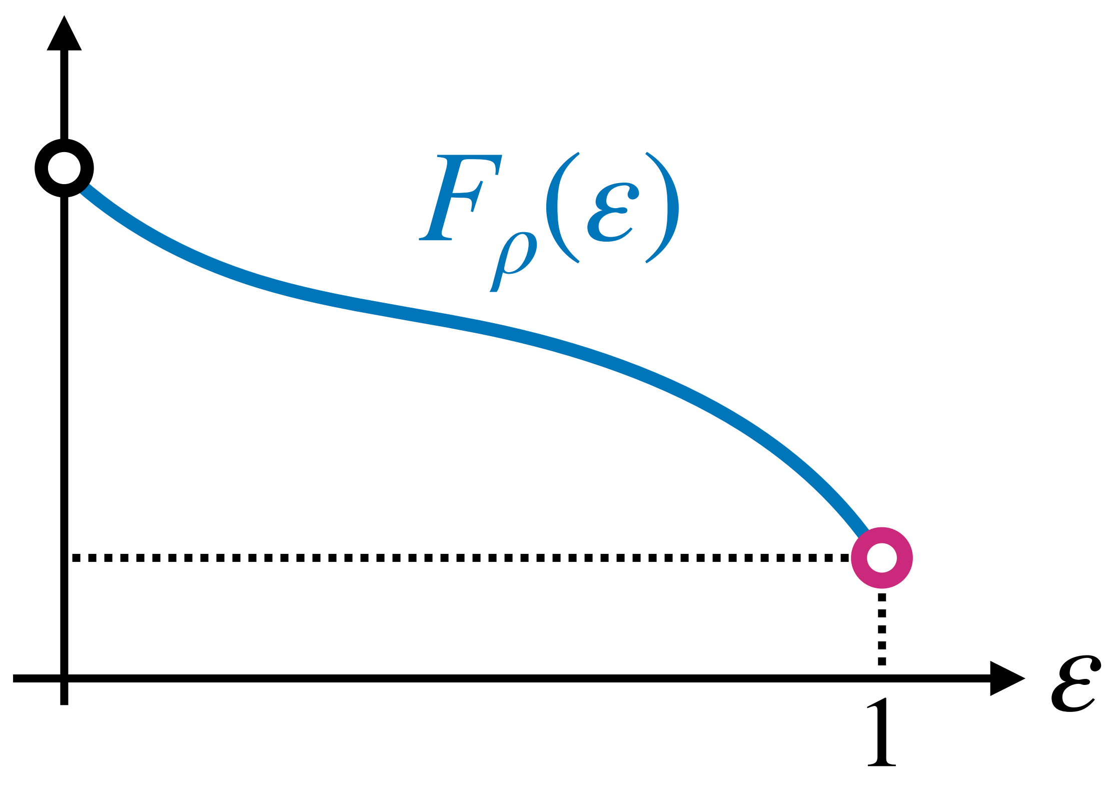
Step 1: Smoothed max-relative entropy

$2^\lambda\sigma - \rho'$ is positive semi-definite

Smoothed max-relative entropy: $D^\varepsilon_{\max}(\rho\|\sigma) := \min_{\rho' \approx_\varepsilon \rho} \inf\{\lambda : \rho' \leq 2^\lambda \sigma\}$

trace-norm ball

Regularisation: $F_\rho(\varepsilon) := \liminf_{n\to\infty} \frac{1}{n} \min_{\sigma_n \in \mathcal{S}_n} D^\varepsilon_{\max}(\rho^{\otimes n} \| \sigma_n)$ ⟿ **non-increasing** in ε.



Brandão/Plenio, CMP 2010; Datta, Int. J. Quantum Inf. 2009; Hayashi/Tomamichel, IEEE 2013.

Step 1: Smoothed max-relative entropy

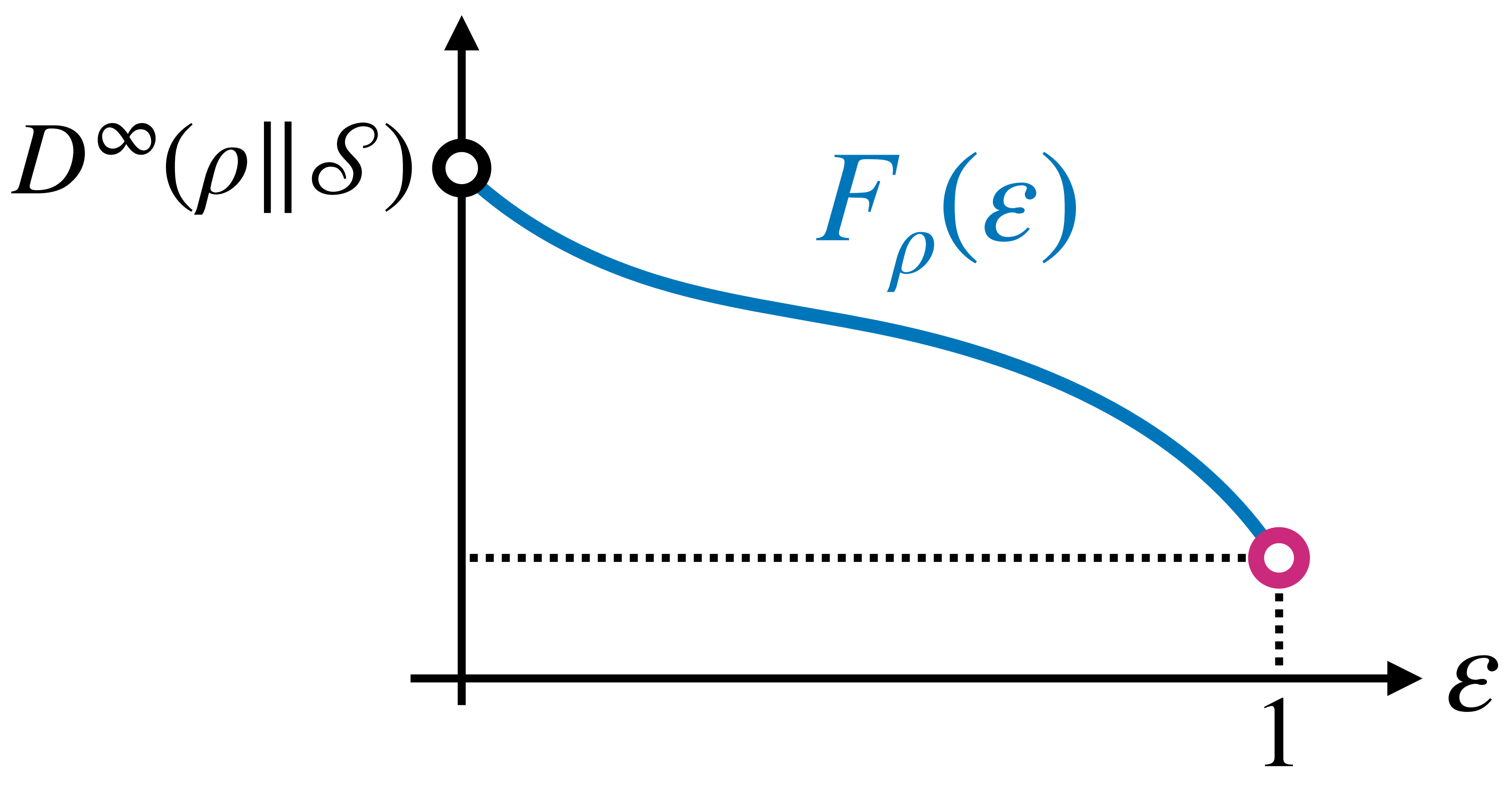
$2^\lambda \sigma - \rho'$ is positive semi-definite

Smoothed max-relative entropy: $D^\varepsilon_{\max}(\rho \| \sigma) := \min_{\rho' \approx_\varepsilon \rho} \inf\{\lambda : \rho' \leq 2^\lambda \sigma\}$

trace-norm ball

Regularisation: $F_\rho(\varepsilon) := \liminf_{n\to\infty} \frac{1}{n} \min_{\sigma_n \in \mathcal{S}_n} D^\varepsilon_{\max}(\rho^{\otimes n} \| \sigma_n)$ ⇝ **non-increasing** in ε.

Fact of life #1: $D^\infty(\rho \| \mathcal{S}) = \lim_{\varepsilon \to 0^+} F_\rho(\varepsilon)$



Brandão/Plenio, CMP 2010; Datta, Int. J. Quantum Inf. 2009; Hayashi/Tomamichel, IEEE 2013.

Step 1: Smoothed max-relative entropy

$2^\lambda \sigma - \rho'$ is positive semi-definite

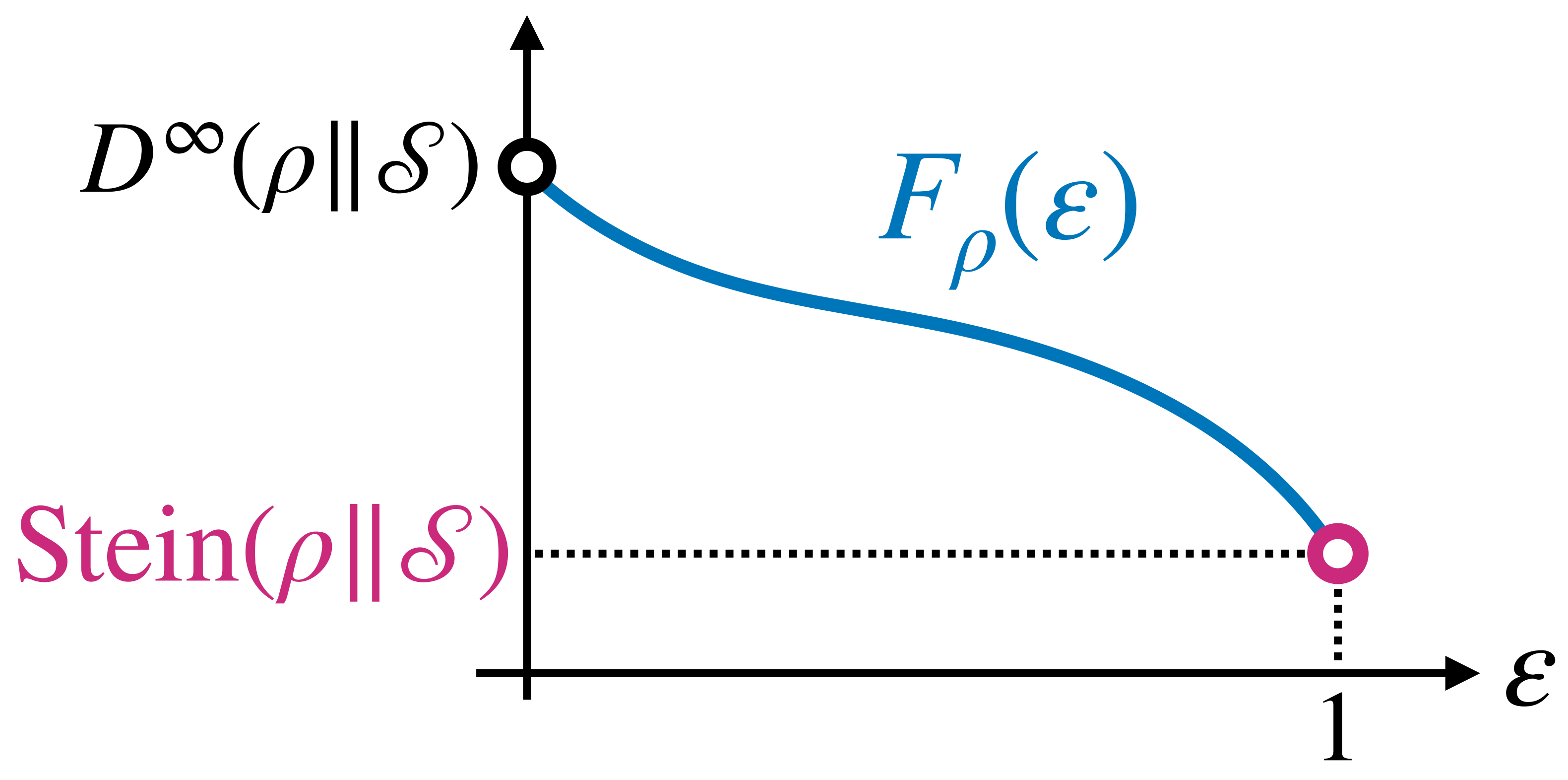
Smoothed max-relative entropy: $D^\varepsilon_{\max}(\rho\|\sigma) := \min_{\rho' \approx_\varepsilon \rho} \inf\{\lambda : \ \rho' \leq 2^\lambda \sigma\}$

trace-norm ball

Regularisation: $F_\rho(\varepsilon) := \liminf_{n\to\infty} \frac{1}{n} \min_{\sigma_n \in \mathcal{S}_n} D^\varepsilon_{\max}(\rho^{\otimes n} \| \sigma_n)$ ⟿ **non-increasing** in ε.

Fact of life #1: $D^\infty(\rho\|\mathcal{S}) = \lim_{\varepsilon\to 0^+} F_\rho(\varepsilon)$

Fact of life #2: $\mathrm{Stein}(\rho\|\mathcal{S}) = \lim_{\varepsilon\to 1^-} F_\rho(\varepsilon)$



Brandão/Plenio, CMP 2010; Datta, Int. J. Quantum Inf. 2009; Hayashi/Tomamichel, IEEE 2013.

Step 1: Smoothed max-relative entropy

$2^\lambda \sigma - \rho'$ is positive semi-definite

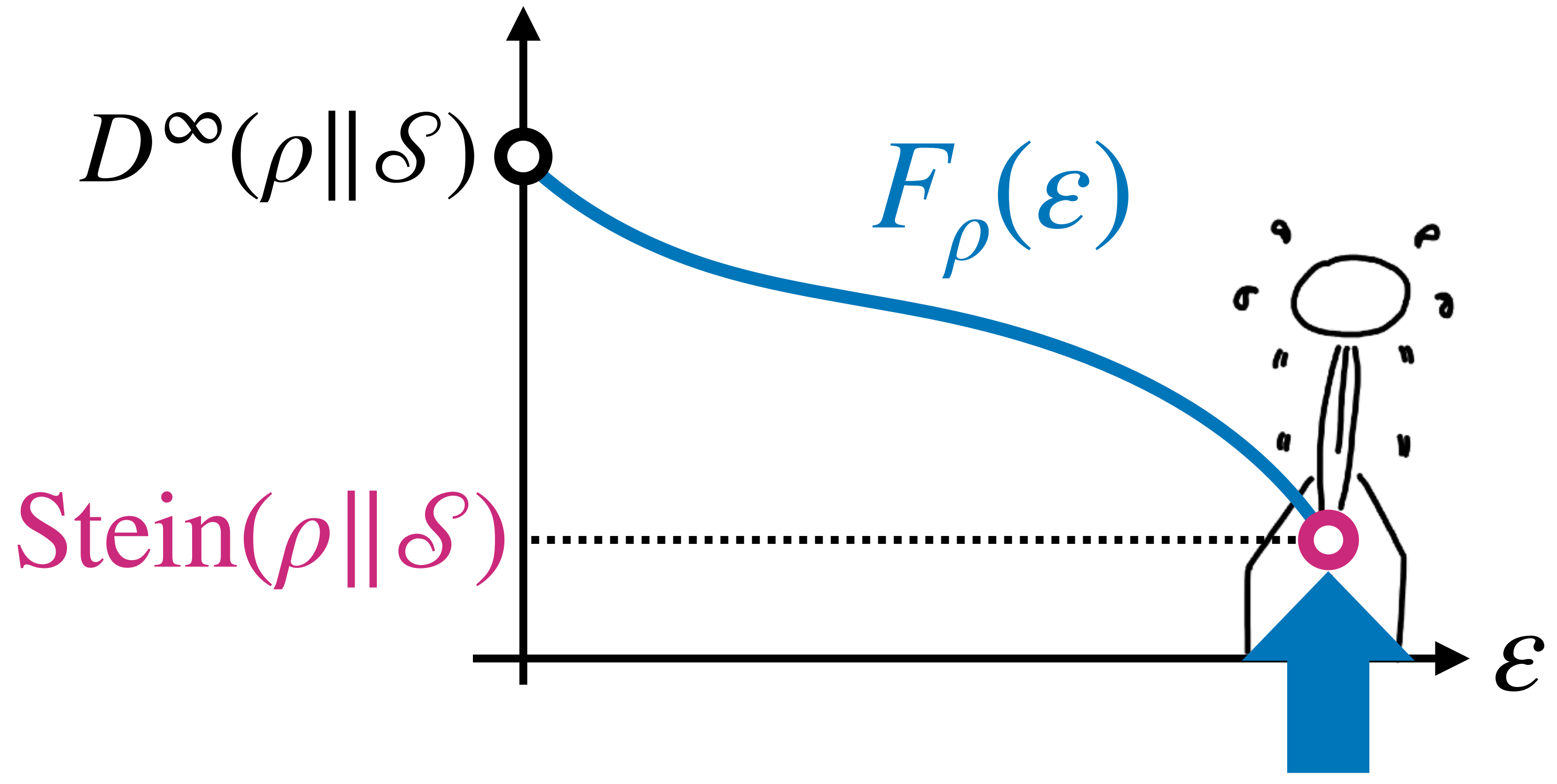
Smoothed max-relative entropy: $D^\varepsilon_{\max}(\rho\|\sigma) := \min_{\rho' \approx_\varepsilon \rho} \inf\{\lambda : \rho' \leq 2^\lambda \sigma\}$

trace-norm ball

Regularisation: $F_\rho(\varepsilon) := \liminf_{n\to\infty} \frac{1}{n} \min_{\sigma_n \in \mathcal{S}_n} D^\varepsilon_{\max}(\rho^{\otimes n} \| \sigma_n)$ ⇝ **non-increasing** in ε.

Fact of life #1: $D^\infty(\rho\|\mathcal{S}) = \lim_{\varepsilon\to 0^+} F_\rho(\varepsilon)$

Fact of life #2: $\mathrm{Stein}(\rho\|\mathcal{S}) = \lim_{\varepsilon\to 1^-} F_\rho(\varepsilon)$



Brandão/Plenio, CMP 2010; Datta, Int. J. Quantum Inf. 2009; Hayashi/Tomamichel, IEEE 2013.

Step 1: Smoothed max-relative entropy

$2^{\lambda}\sigma - \rho'$ is positive semi-definite

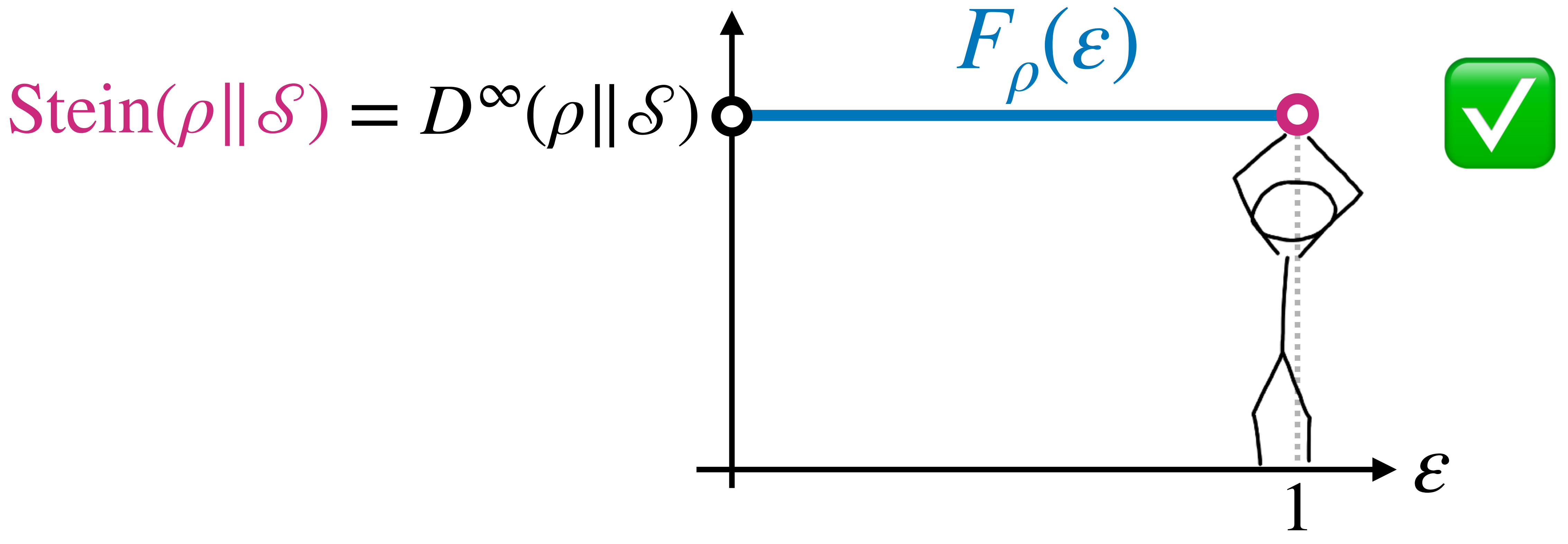
Smoothed max-relative entropy: $D^{\varepsilon}_{\max}(\rho\|\sigma) := \min_{\rho' \approx_\varepsilon \rho} \inf\{\lambda:\ \rho' \leq 2^{\lambda}\sigma\}$

trace-norm ball

Regularisation: $F_\rho(\varepsilon) := \liminf_{n\to\infty} \frac{1}{n} \min_{\sigma_n \in \mathcal{S}_n} D^{\varepsilon}_{\max}(\rho^{\otimes n} \| \sigma_n)$ ⇝ **non-increasing** in ε.

Fact of life #1: $D^{\infty}(\rho\|\mathcal{S}) = \lim_{\varepsilon\to 0^+} F_\rho(\varepsilon)$

Fact of life #2: $\mathrm{Stein}(\rho\|\mathcal{S}) = \lim_{\varepsilon\to 1^-} F_\rho(\varepsilon)$



Brandão/Plenio, CMP 2010; Datta, Int. J. Quantum Inf. 2009; Hayashi/Tomamichel, IEEE 2013.

Step 2: Brandão/Plenio axioms

Abstract even further.

E.g. separable states

Hilbert space $\mathscr{H}$. For all n, set of "easy-to-prepare" ***free states*** $\mathscr{F}_n \subseteq \mathscr{D}\left(\mathscr{H}^{\otimes n}\right)$.

Step 2: Brandão/Plenio axioms

Abstract even further.

E.g. separable states

Hilbert space $\mathscr{H}$. For all *n*, set of "easy-to-prepare" ***free states*** $\mathscr{F}_n \subseteq \mathscr{D}\left(\mathscr{H}^{\otimes n}\right)$.

1. $\mathscr{F}_n$ closed and convex.
2. $\mathscr{F}_1$ contains a full-rank state.

Step 2: Brandão/Plenio axioms

Abstract even further.

E.g. separable states

Hilbert space $\mathscr{H}$. For all *n*, set of "easy-to-prepare" ***free states*** $\mathscr{F}_n \subseteq \mathscr{D}\left(\mathscr{H}^{\otimes n}\right)$.

1. $\mathscr{F}_n$ closed and convex.
2. $\mathscr{F}_1$ contains a full-rank state.
3. $(\mathscr{F}_n)_n$ closed under partial trace: $\mathrm{Tr}_m \mathscr{F}_{n+m} \subseteq \mathscr{F}_n$.
4. $(\mathscr{F}_n)_n$ closed under tensor products: $\mathscr{F}_n \otimes \mathscr{F}_m \subseteq \mathscr{F}_{n+m}$.
5. $\mathscr{F}_n$ closed under permutations: $U_\pi \mathscr{F}_n U_\pi^\dagger \subseteq \mathscr{F}_n$

Classical version

We can now apply the framework also to *classical* resource theories!

Alphabet $\mathscr{X}$. For all n, $\mathscr{F}_n$:= {*free probability distributions* on $\mathscr{X}^n$}. BP axioms.

Classical version

We can now apply the framework also to *classical* resource theories!

Alphabet $\mathscr{X}$. For all n, $\mathscr{F}_n$:= {*free probability distributions* on $\mathscr{X}^n$}. BP axioms.

To prove: $$F_p(\varepsilon) = \liminf_{n\to\infty} \frac{1}{n} \min_{q_n \in \mathscr{F}_n} D^{\varepsilon}_{\max}(p^{\otimes n} \,\|\, q_n) = D^{\infty}(p\|\mathscr{F}) \quad \forall\, \varepsilon \in (0,1)\,.$$

Classical version

We can now apply the framework also to *classical* resource theories!

Alphabet $\mathscr{X}$. For all n, $\mathscr{F}_n$:= {*free probability distributions* on $\mathscr{X}^n$}. BP axioms.

To prove: $F_p(\varepsilon) = \liminf_{n\to\infty} \frac{1}{n} \min_{q_n \in \mathscr{F}_n} D^{\varepsilon}_{\max}(p^{\otimes n} \,\|\, q_n) = D^{\infty}(p\|\mathscr{F}) \quad \forall\, \varepsilon \in (0,1)\,.$

By contradiction: if LHS ≤ λ < RHS there are sequences

$$p_n \approx_\varepsilon p^{\otimes n}, \quad q_n \in \mathscr{F}_n : \qquad p_n \leq 2^{n\lambda} q_n \qquad (\varepsilon \text{ fixed, not small})$$

Classical version

We can now apply the framework also to *classical* resource theories!

Alphabet $\mathcal{X}$. For all n, $\mathcal{F}_n$:= {*free probability distributions* on $\mathcal{X}^n$}. BP axioms.

To prove: $$F_p(\varepsilon) = \liminf_{n\to\infty} \frac{1}{n} \min_{q_n \in \mathcal{F}_n} D^{\varepsilon}_{\max}(p^{\otimes n} \| q_n) = D^{\infty}(p\|\mathcal{F}) \quad \forall\, \varepsilon \in (0,1)\,.$$

By contradiction: if LHS ≤ λ < RHS there are sequences

$$p_n \approx_\varepsilon p^{\otimes n}, \quad q_n \in \mathcal{F}_n : \qquad p_n \leq 2^{n\lambda} q_n \qquad (\varepsilon \text{ fixed, not small})$$

Note: by Axiom 5, p_n, q_n can be taken to be permutationally symmetric wlog.

Classical version

We can now apply the framework also to *classical* resource theories!

Alphabet $\mathscr{X}$. For all n, $\mathscr{F}_n$:= {*free probability distributions* on $\mathscr{X}^n$}. BP axioms.

To prove: $$F_p(\varepsilon) = \liminf_{n\to\infty} \frac{1}{n} \min_{q_n \in \mathscr{F}_n} D^{\varepsilon}_{\max}(p^{\otimes n} \,\|\, q_n) = D^{\infty}(p\|\mathscr{F}) \quad \forall\, \varepsilon \in (0,1)\,.$$

By contradiction: if LHS $\leq \lambda <$ RHS there are sequences

$$p_n \approx_\varepsilon p^{\otimes n}, \quad q_n \in \mathscr{F}_n : \qquad p_n \leq 2^{n\lambda} q_n$$ (ε fixed, not small)

Note: by Axiom 5, p_n, q_n can be taken to be permutationally symmetric wlog.

$\Longrightarrow$ $p_n(x^n)$ only depends on the **type** of x^n, same for q_n .

Classical version

We can now apply the framework also to *classical* resource theories!

Alphabet $\mathcal{X}$. For all n, $\mathcal{F}_n$:= {*free probability distributions* on $\mathcal{X}^n$}. BP axioms.

To prove: $$F_p(\varepsilon) = \liminf_{n\to\infty} \frac{1}{n} \min_{q_n \in \mathcal{F}_n} D^{\varepsilon}_{\max}(p^{\otimes n} \,\|\, q_n) = D^{\infty}(p\|\mathcal{F}) \quad \forall\, \varepsilon \in (0,1)\,.$$

By contradiction: if LHS $\leq \lambda <$ RHS there are sequences

$$p_n \approx_\varepsilon p^{\otimes n}, \quad q_n \in \mathcal{F}_n : \qquad p_n \leq 2^{n\lambda} q_n \qquad (\varepsilon \text{ fixed, not small})$$

Note: by Axiom 5, p_n, q_n can be taken to be permutationally symmetric wlog.

$\Longrightarrow$ $p_n(x^n)$ only depends on the type of x^n, same for q_n .

???

Hands-on introduction to the theory of types

Def. The **type** t_{x^n} of a sequence $x^n \in \mathscr{X}^n$ is its empirical probability distribution:

$$t_{x^n} : \mathscr{X} \longrightarrow [0,1]\,, \qquad t_{x^n}(x) := \frac{\text{\# times } x \text{ appears in } x^n}{n}$$

Csiszár and Körner, Information theory: coding theorems for discrete memoryless systems (1981).

Hands-on introduction to the theory of types

Def. The **type** t_{x^n} of a sequence $x^n \in \mathscr{X}^n$ is its empirical probability distribution:

$$t_{x^n} : \mathscr{X} \longrightarrow [0,1]\,, \qquad t_{x^n}(x) := \frac{\text{\# times } x \text{ appears in } x^n}{n}$$

Fact #1: Law of large numbers $\Rightarrow p^{\otimes n}$ concentrated around type p.

Csiszár and Körner, Information theory: coding theorems for discrete memoryless systems (1981).

Hands-on introduction to the theory of types

Def. The **type** t_{x^n} of a sequence $x^n \in \mathscr{X}^n$ is its empirical probability distribution:

$$t_{x^n} : \mathscr{X} \longrightarrow [0,1]\,, \qquad t_{x^n}(x) := \frac{\text{\# times } x \text{ appears in } x^n}{n}$$

Fact #1: Law of large numbers $\Rightarrow p^{\otimes n}$ concentrated around type p.

Fact #2: Exp many sequences ($|\mathscr{X}|^n$), but only poly many types ($\leq (n+1)^{|\mathscr{X}|}$).

Csiszár and Körner, Information theory: coding theorems for discrete memoryless systems (1981).

Hands-on introduction to the theory of types

Def. The **type** t_{x^n} of a sequence $x^n \in \mathscr{X}^n$ is its empirical probability distribution:

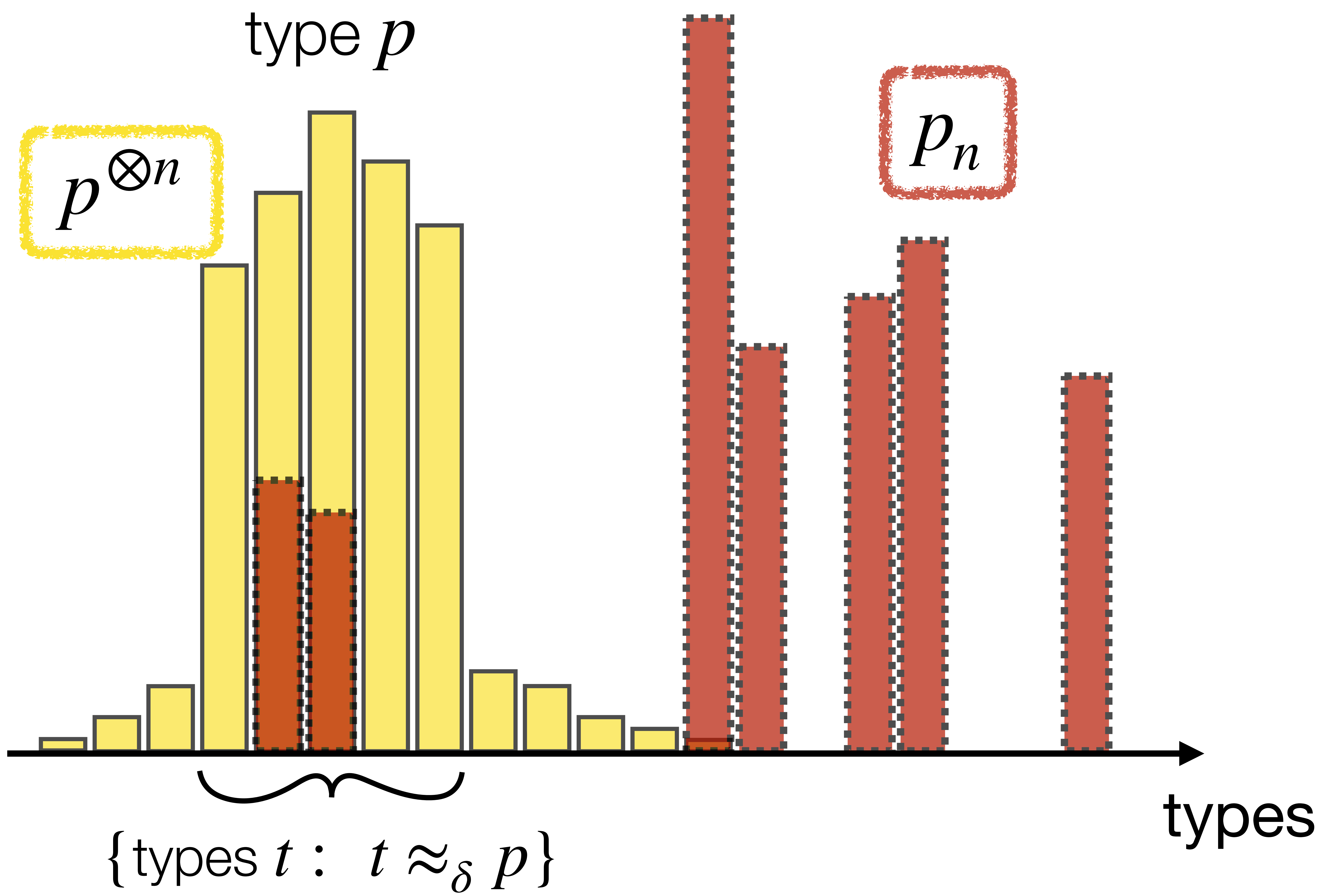
$$t_{x^n} : \mathscr{X} \longrightarrow [0,1] , \qquad t_{x^n}(x) := \frac{\text{\# times } x \text{ appears in } x^n}{n}$$

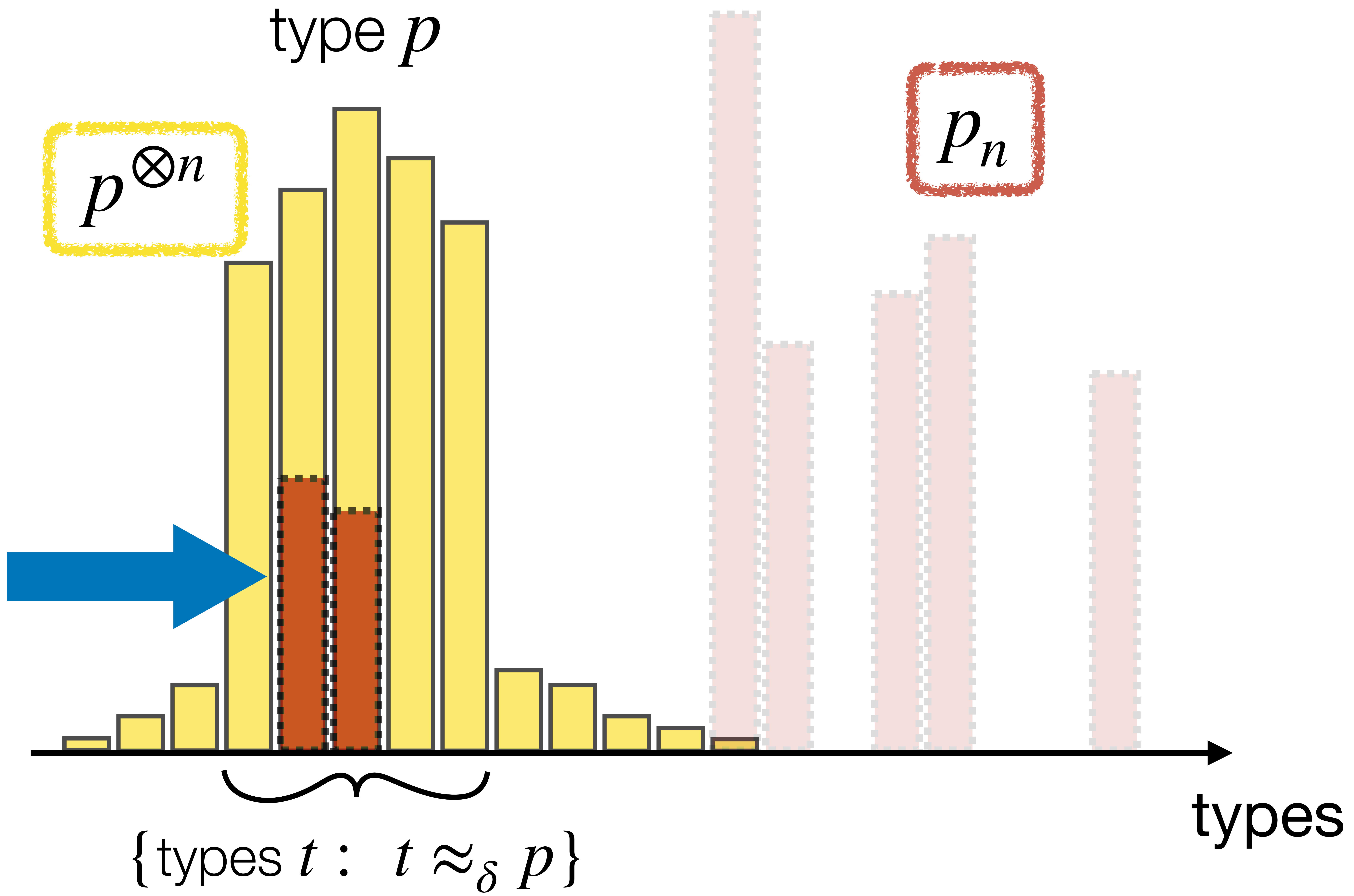
Fact #1: Law of large numbers $\Rightarrow p^{\otimes n}$ concentrated around type p.

Fact #2: Exp many sequences ($|\mathscr{X}|^n$), but only poly many types ($\leq (n+1)^{|\mathscr{X}|}$).

$\longrightarrow$ Represent every symmetric p_n, q_n in **type space** instead of **sequence space.**

Csiszár and Körner, Information theory: coding theorems for discrete memoryless systems (1981).


type p
$p^{\otimes n}$
p_n
types
$\{\text{types } t : \; t \approx_\delta p\}$



type p
$p^{\otimes n}$
p_n
$\{\text{types } t : \ t \approx_\delta p\}$
types

Step 3: *Blurring*

Step 3: *Blurring*

Def. Fix n, m (think $m \sim \delta n$, δ small), the **blurring map** $B_{n,m}$ is

$$B_{n,m} : \mathscr{P}(\mathscr{X}^n) \longrightarrow \mathscr{P}(\mathscr{X}^n)$$

$$B_{n,m}(r_n) := \mathrm{tr}_m\left[\mathrm{sym}_{n+m}\left(r_n \otimes q_0^{\otimes m}\right)\right]$$

Step 3: *Blurring*

Def. Fix n, m (think $m \sim \delta n$, δ small), the **blurring map** $B_{n,m}$ is

$$B_{n,m} : \mathscr{P}\left(\mathscr{X}^n\right) \longrightarrow \mathscr{P}\left(\mathscr{X}^n\right)$$

$$B_{n,m}(r_n) := \mathrm{tr}_m\left[\mathrm{sym}_{n+m}\left(r_n \otimes q_0^{\otimes m}\right)\right]$$

full-support $q_0 \in \mathscr{F}_1$

Step 3: *Blurring*

Def. Fix n, m (think $m \sim \delta n$, δ small), the **blurring map** $B_{n,m}$ is

$$B_{n,m} : \mathscr{P}(\mathscr{X}^n) \longrightarrow \mathscr{P}(\mathscr{X}^n)$$

$$B_{n,m}(r_n) := \mathrm{tr}_m\left[\mathrm{sym}_{n+m}\left(r_n \otimes q_0^{\otimes m}\right)\right]$$

apply a random permutation

full-support $q_0 \in \mathscr{F}_1$

Step 3: *Blurring*

Def. Fix n, m (think $m \sim \delta n$, δ small), the **blurring map** $B_{n,m}$ is

$$B_{n,m} : \mathscr{P}(\mathscr{X}^n) \longrightarrow \mathscr{P}(\mathscr{X}^n)$$

$$B_{n,m}(r_n) := \mathrm{tr}_m\left[\mathrm{sym}_{n+m}\left(r_n \otimes q_0^{\otimes m}\right)\right]$$

discard m symbols

apply a random permutation

full-support $q_0 \in \mathscr{F}_1$

Step 3: *Blurring*

Def. Fix n, m (think $m \sim \delta n$, δ small), the **blurring map** $B_{n,m}$ is

$$B_{n,m} : \ \mathscr{P}\big(\mathscr{X}^n\big) \longrightarrow \mathscr{P}\big(\mathscr{X}^n\big)$$

$$B_{n,m}(r_n) := \mathrm{tr}_m\big[\,\mathrm{sym}_{n+m}\big(r_n \otimes \ q_0^{\otimes m}\big)\big]$$

discard m symbols

apply a random permutation

full-support $q_0 \in \mathscr{F}_1$

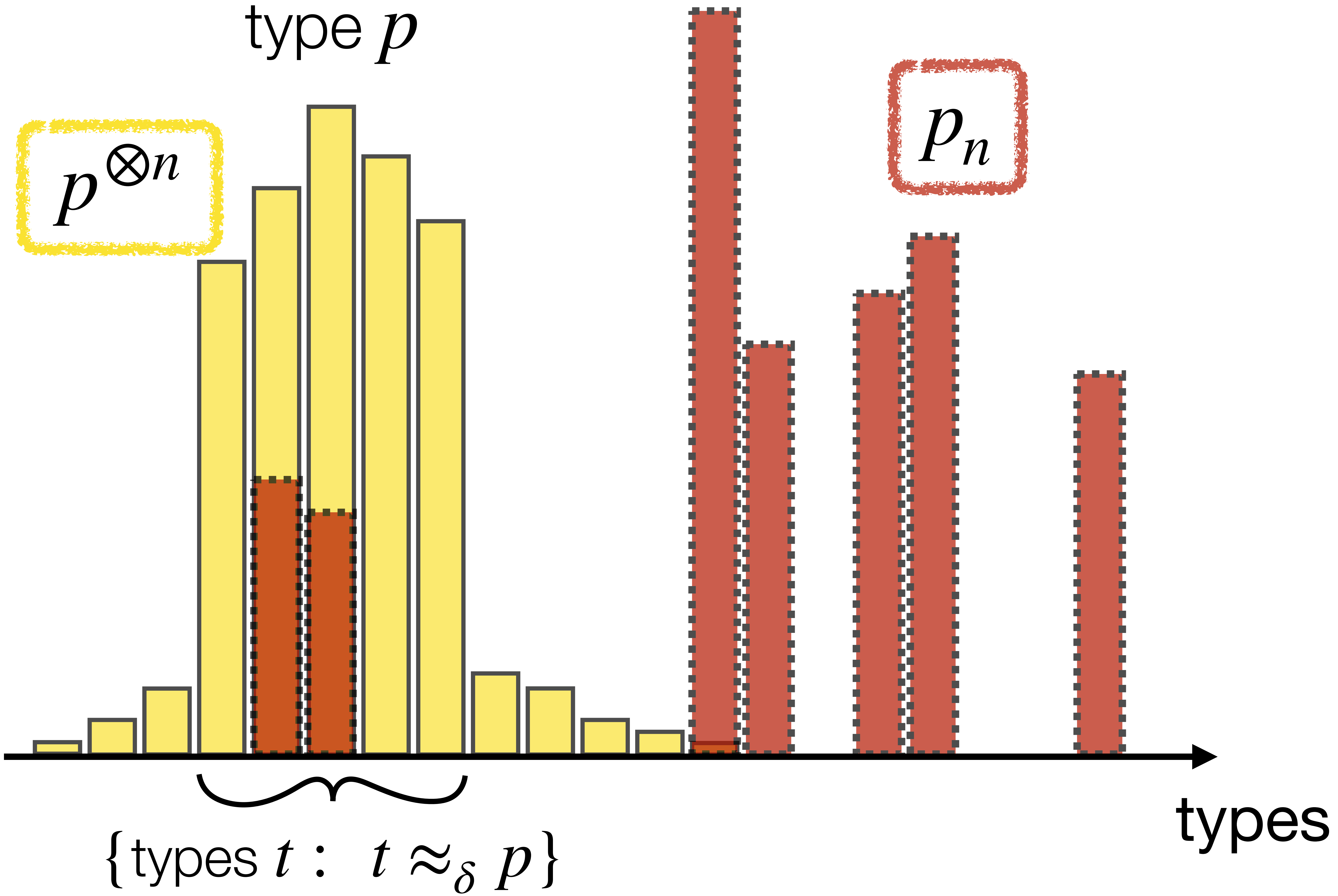
Axioms $\Longrightarrow$ $B_{n,m}$ maps free states to free states!

$$p^{\otimes n} \approx_{\varepsilon} p_n \leq 2^{\lambda n} q_n$$

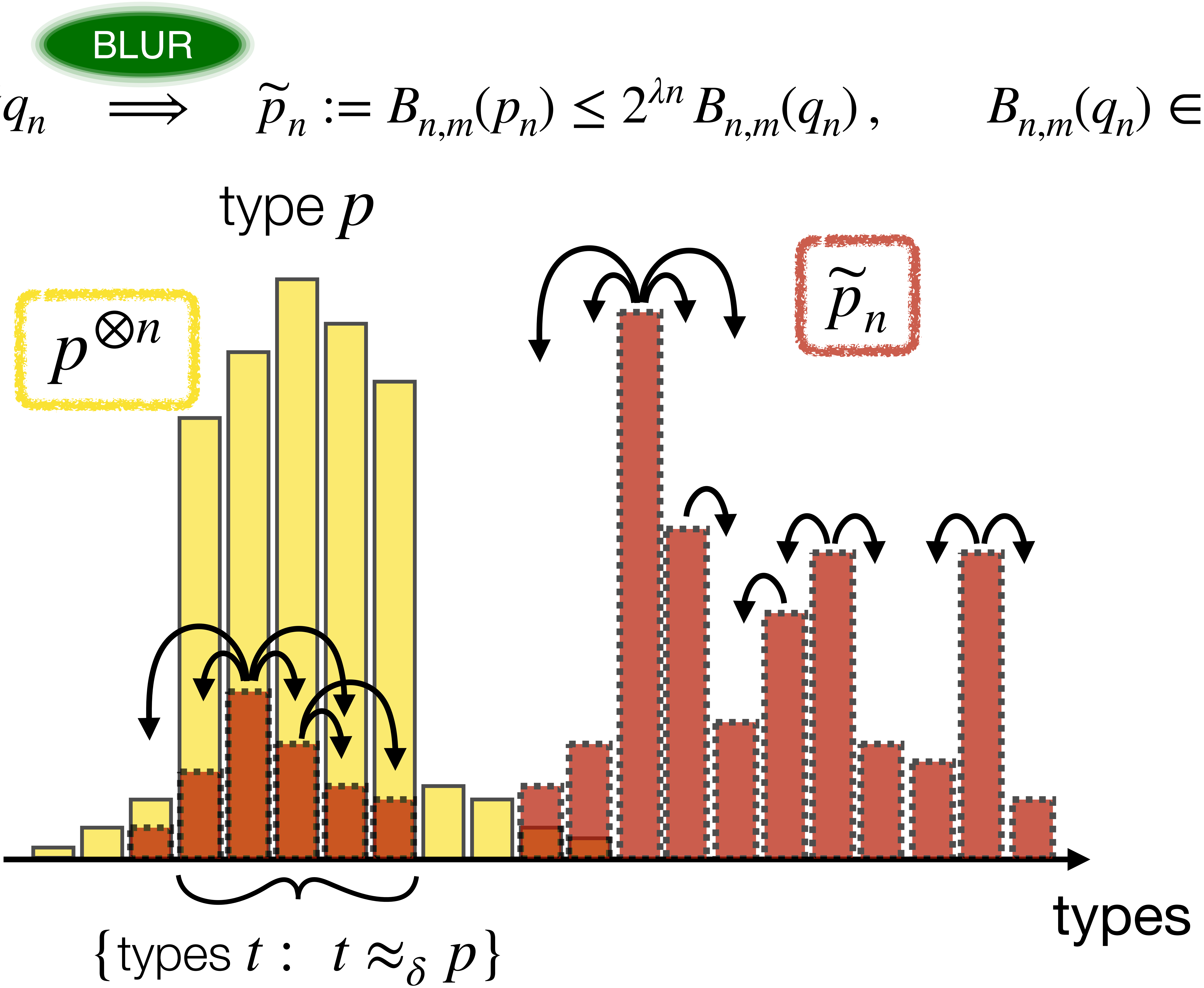
BLUR

$$p^{\otimes n} \approx_\varepsilon p_n \leq 2^{\lambda n} q_n \quad \Longrightarrow \quad \widetilde{p}_n := B_{n,m}(p_n) \leq 2^{\lambda n} B_{n,m}(q_n)\,, \qquad B_{n,m}(q_n) \in \mathscr{F}_n\,.$$

$$p^{\otimes n} \approx_\varepsilon p_n \le 2^{\lambda n} q_n \quad \Longrightarrow \quad \widetilde{p}_n := B_{n,m}(p_n) \le 2^{\lambda n} B_{n,m}(q_n)\,, \qquad B_{n,m}(q_n) \in \mathscr{F}_n\,.$$

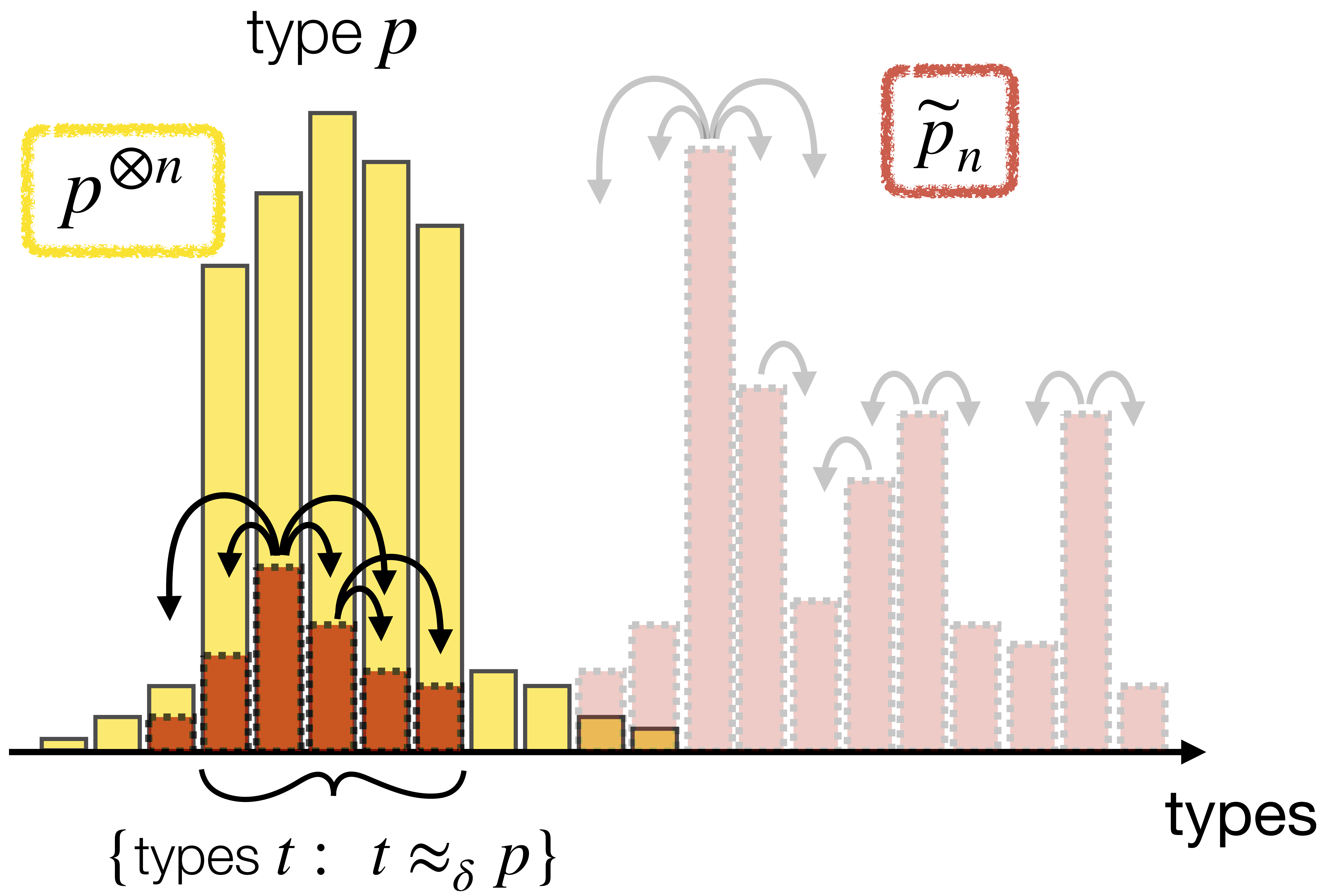


$$p^{\otimes n} \approx_\varepsilon p_n \le 2^{\lambda n} q_n \quad \Longrightarrow \quad \widetilde{p}_n := B_{n,m}(p_n) \le 2^{\lambda n} B_{n,m}(q_n)\,, \qquad B_{n,m}(q_n) \in \mathscr{F}_n\,.$$

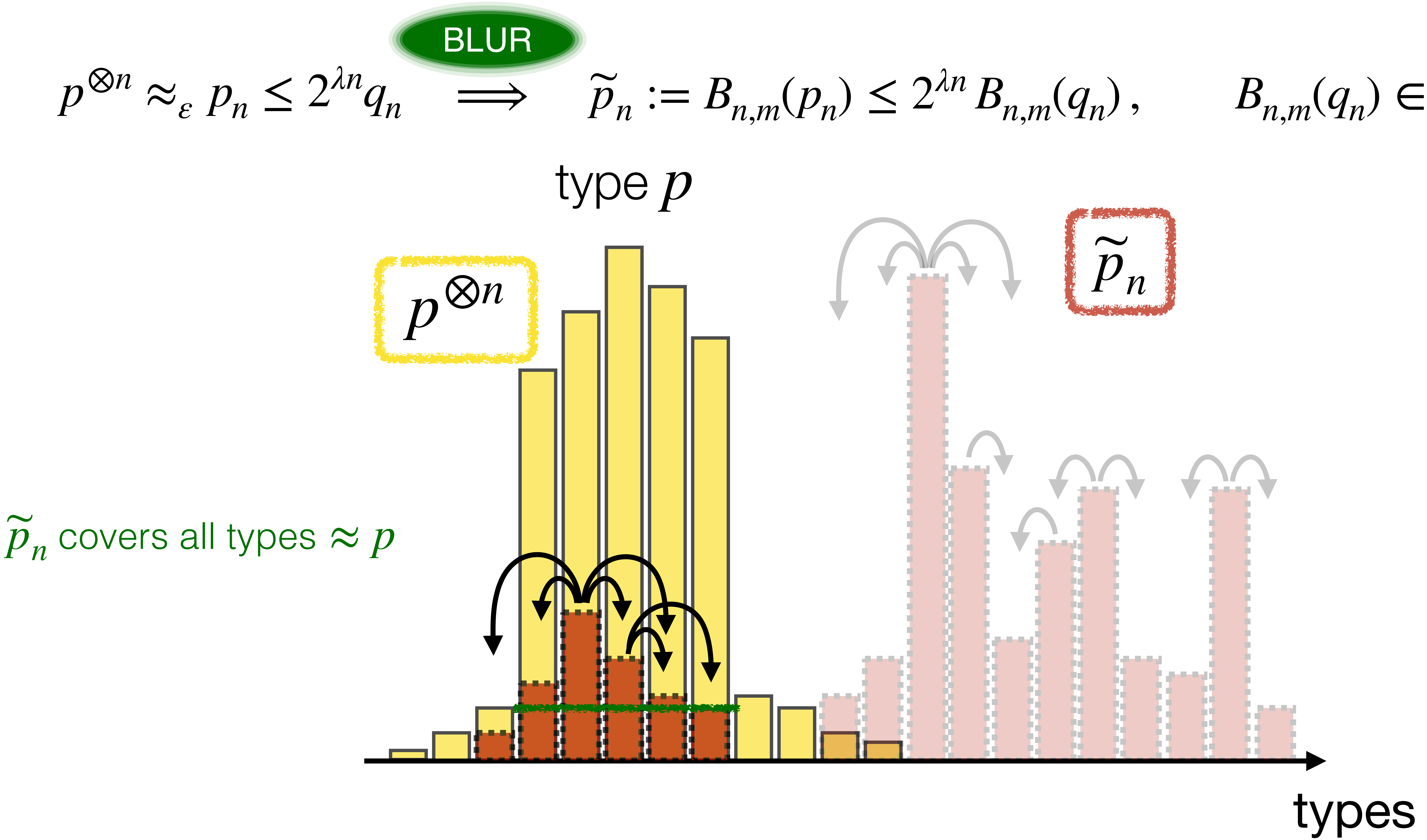


$$p^{\otimes n} \approx_\varepsilon p_n \le 2^{\lambda n} q_n \quad \Longrightarrow \quad \widetilde{p}_n := B_{n,m}(p_n) \le 2^{\lambda n} B_{n,m}(q_n)\,, \qquad B_{n,m}(q_n) \in \mathscr{F}_n\,.$$

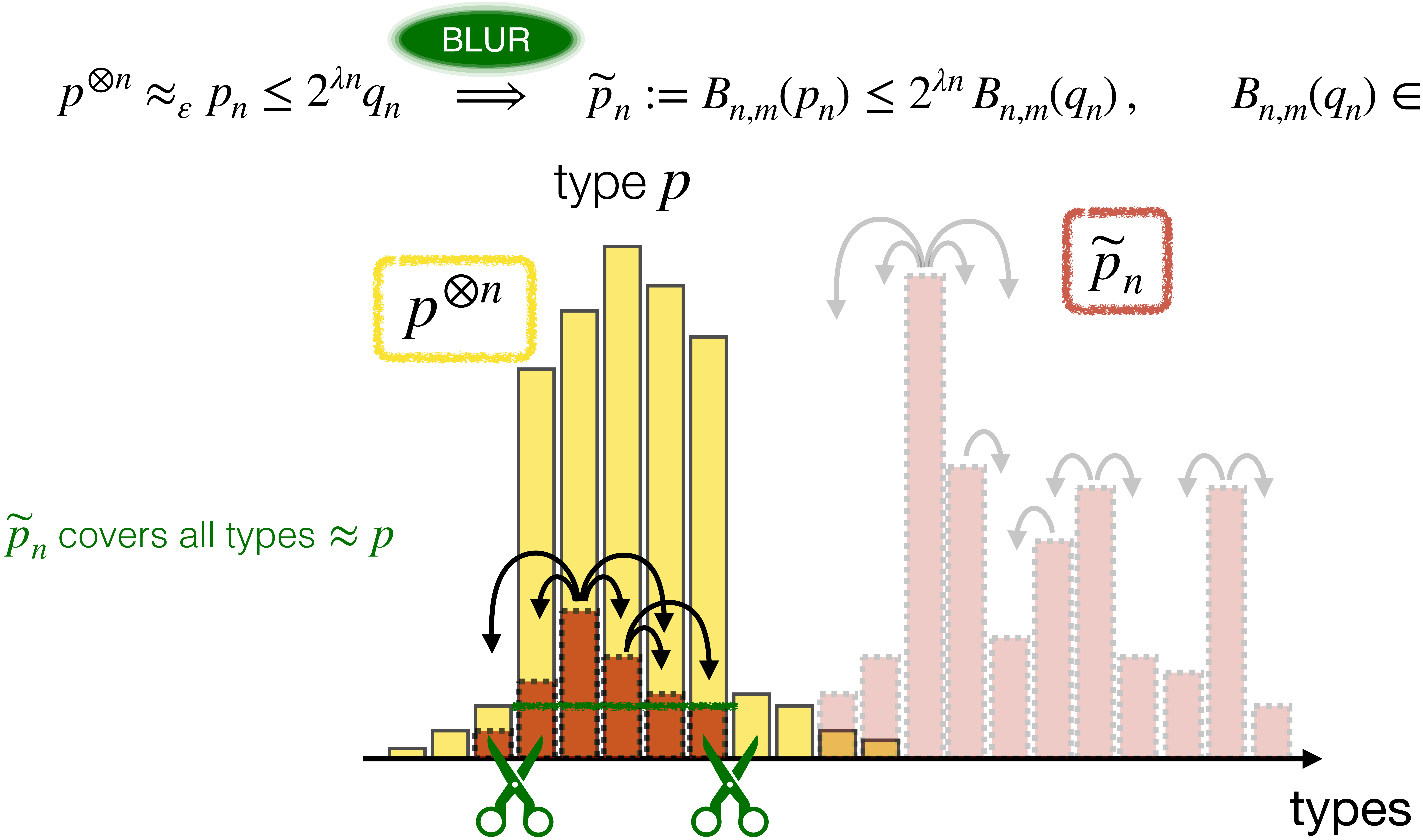
BLUR



$$p^{\otimes n} \approx_{\varepsilon} p_n \leq 2^{\lambda n} q_n \quad \Longrightarrow \quad \widetilde{p}_n := B_{n,m}(p_n) \leq 2^{\lambda n} B_{n,m}(q_n)\,, \qquad B_{n,m}(q_n) \in \mathscr{F}_n\,.$$

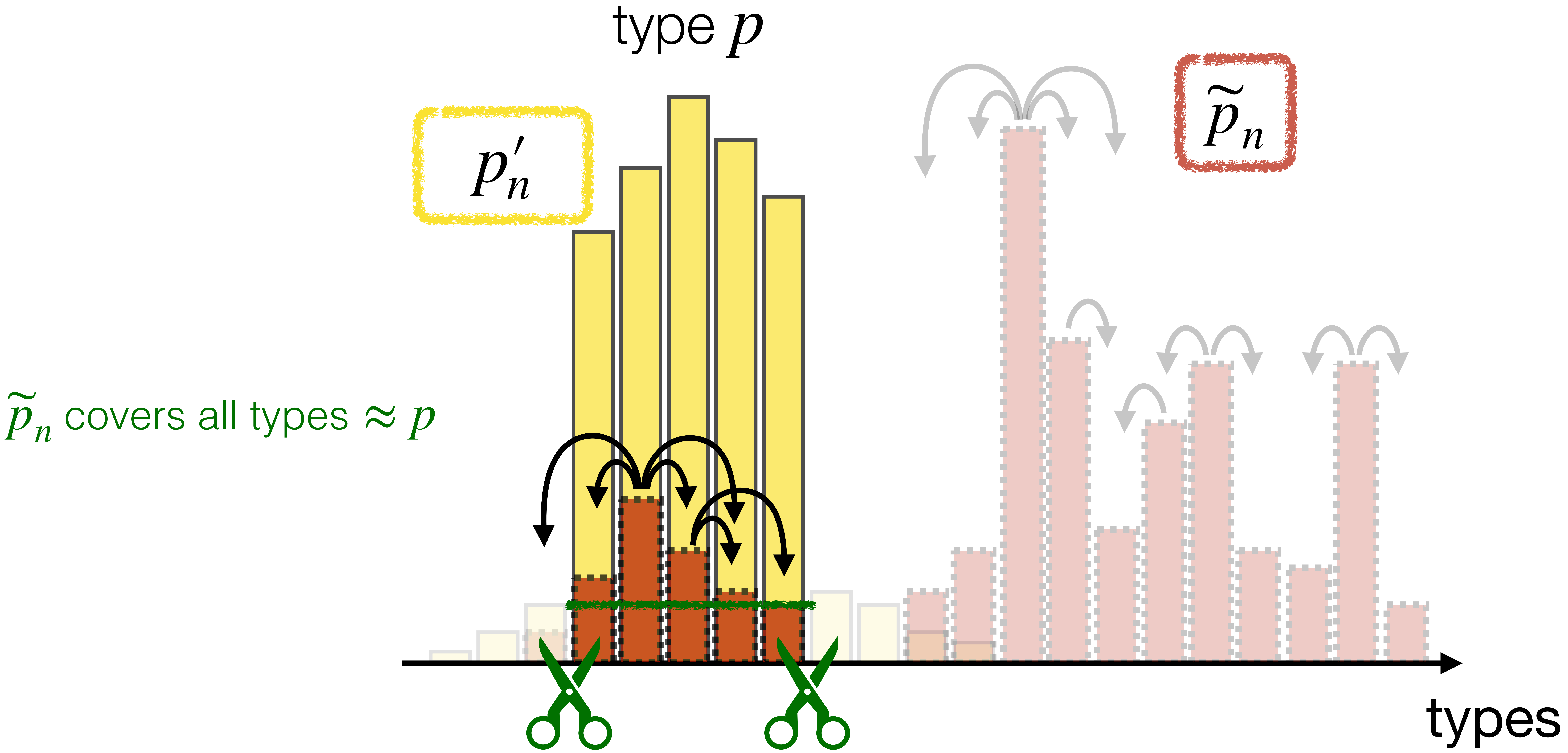


$$p^{\otimes n} \approx_\varepsilon p_n \leq 2^{\lambda n} q_n \quad \Longrightarrow \quad \widetilde{p}_n := B_{n,m}(p_n) \leq 2^{\lambda n} B_{n,m}(q_n)\,, \qquad B_{n,m}(q_n) \in \mathscr{F}_n\,.$$

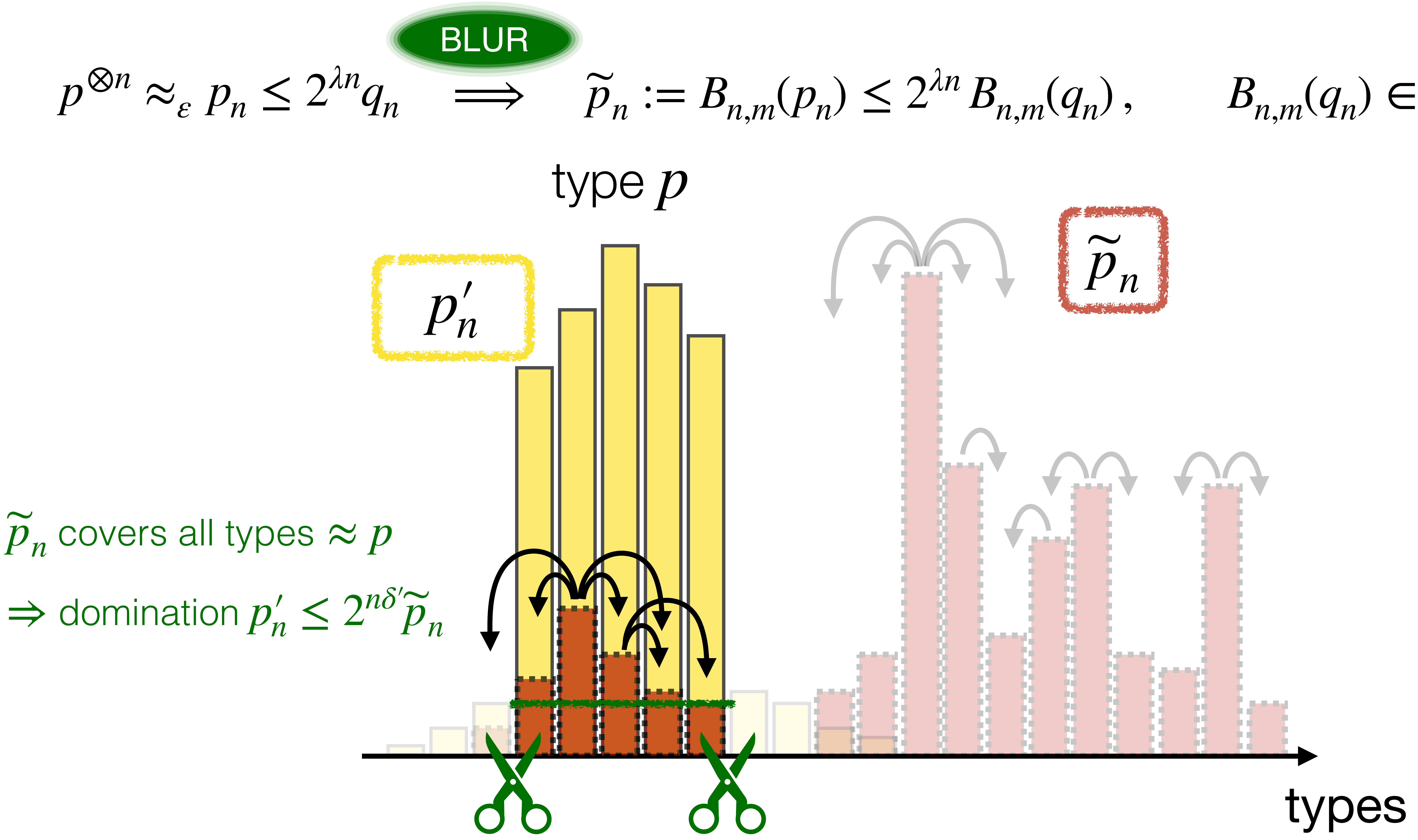


BLUR

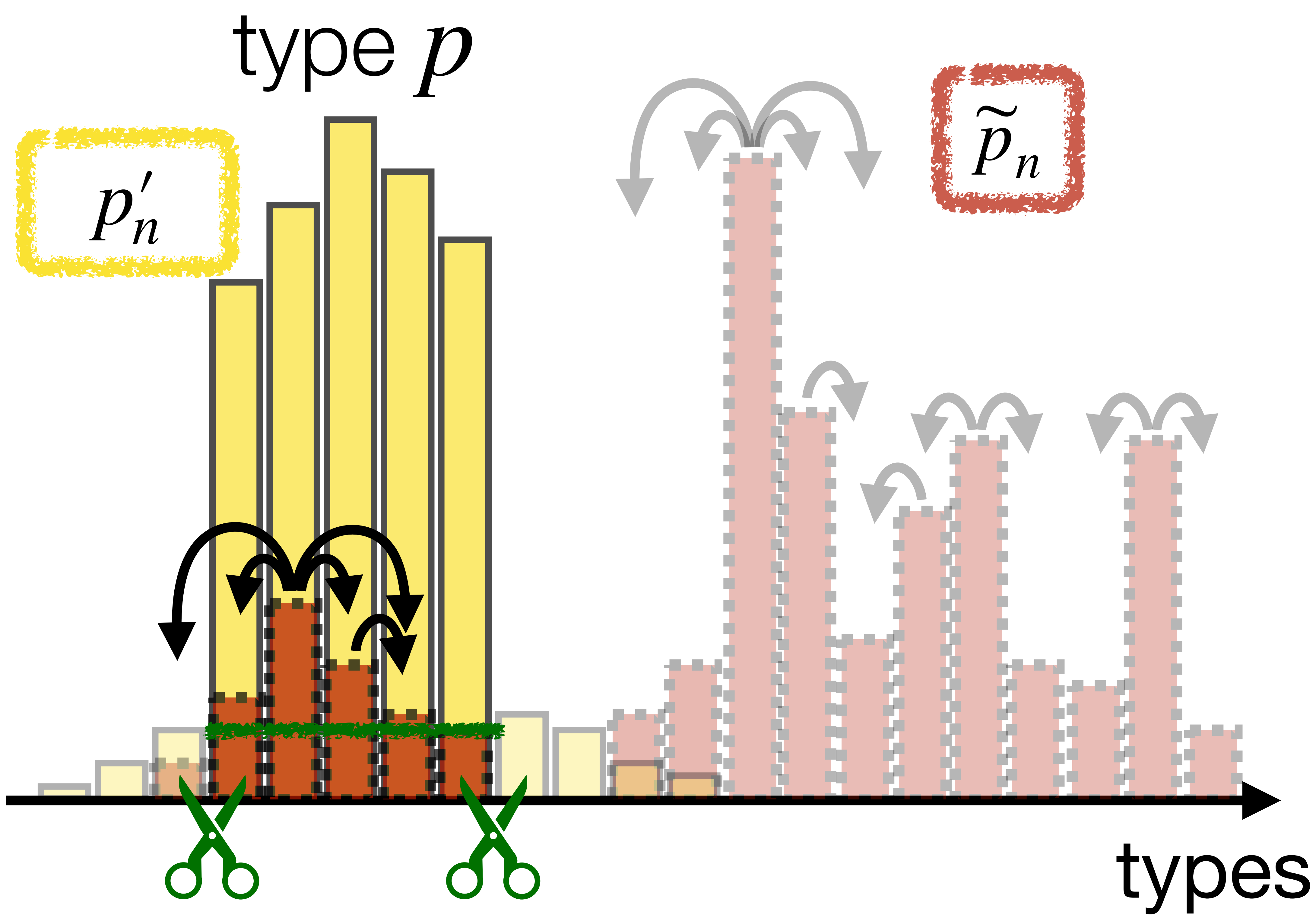
$$p^{\otimes n} \approx_{\varepsilon} p_n \leq 2^{\lambda n} q_n \quad \Longrightarrow \quad \widetilde{p}_n := B_{n,m}(p_n) \leq 2^{\lambda n} B_{n,m}(q_n)\,, \qquad B_{n,m}(q_n) \in \mathscr{F}_n\,.$$



$$p^{\otimes n} \approx_{\varepsilon} p_n \leq 2^{\lambda n} q_n \quad \Longrightarrow \quad \widetilde{p}_n := B_{n,m}(p_n) \leq 2^{\lambda n} B_{n,m}(q_n)\,, \qquad B_{n,m}(q_n) \in \mathscr{F}_n\,.$$

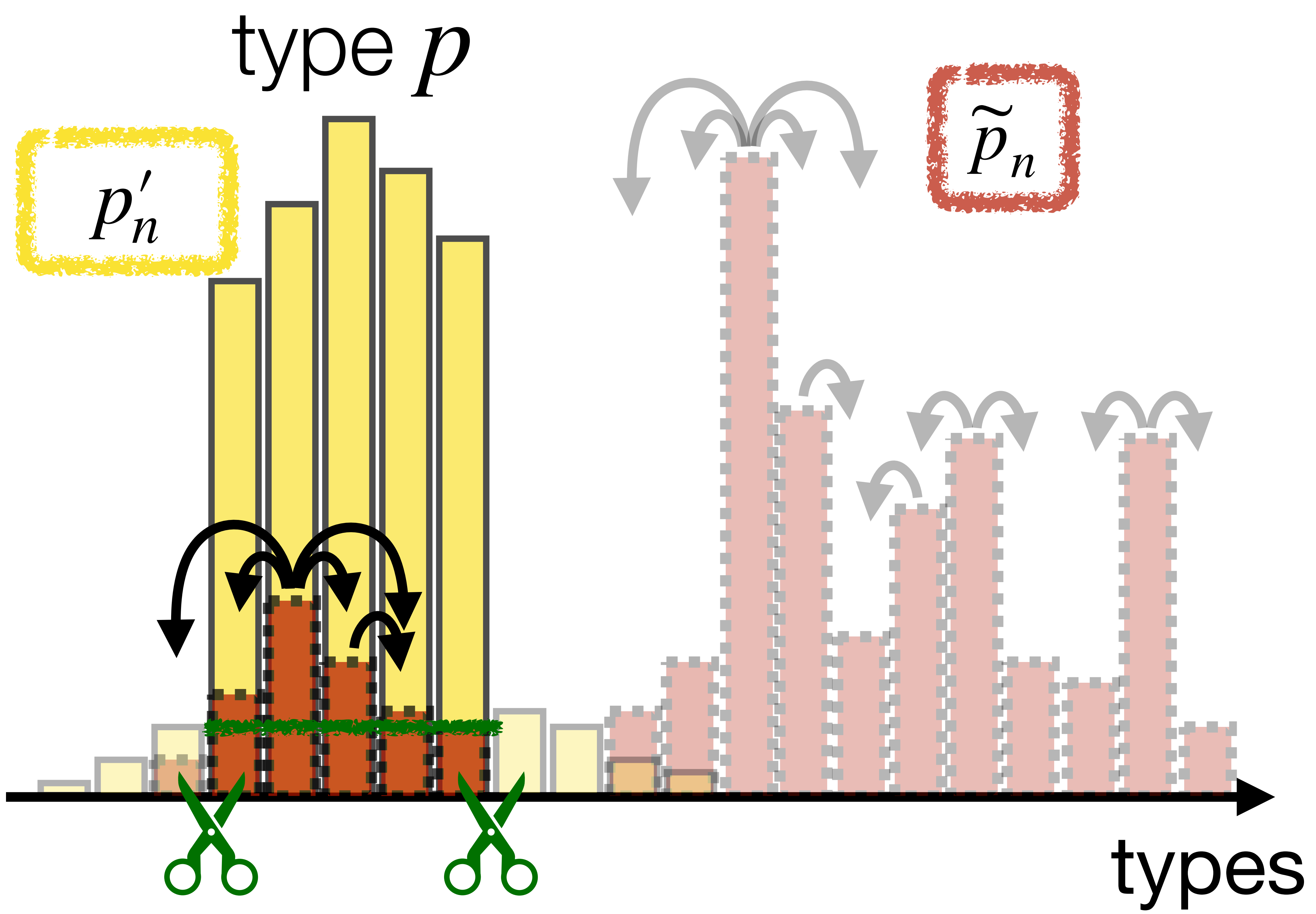


$$p^{\otimes n} \approx_\eta p'_n \leq 2^{n\delta'} \widetilde{p}_n$$



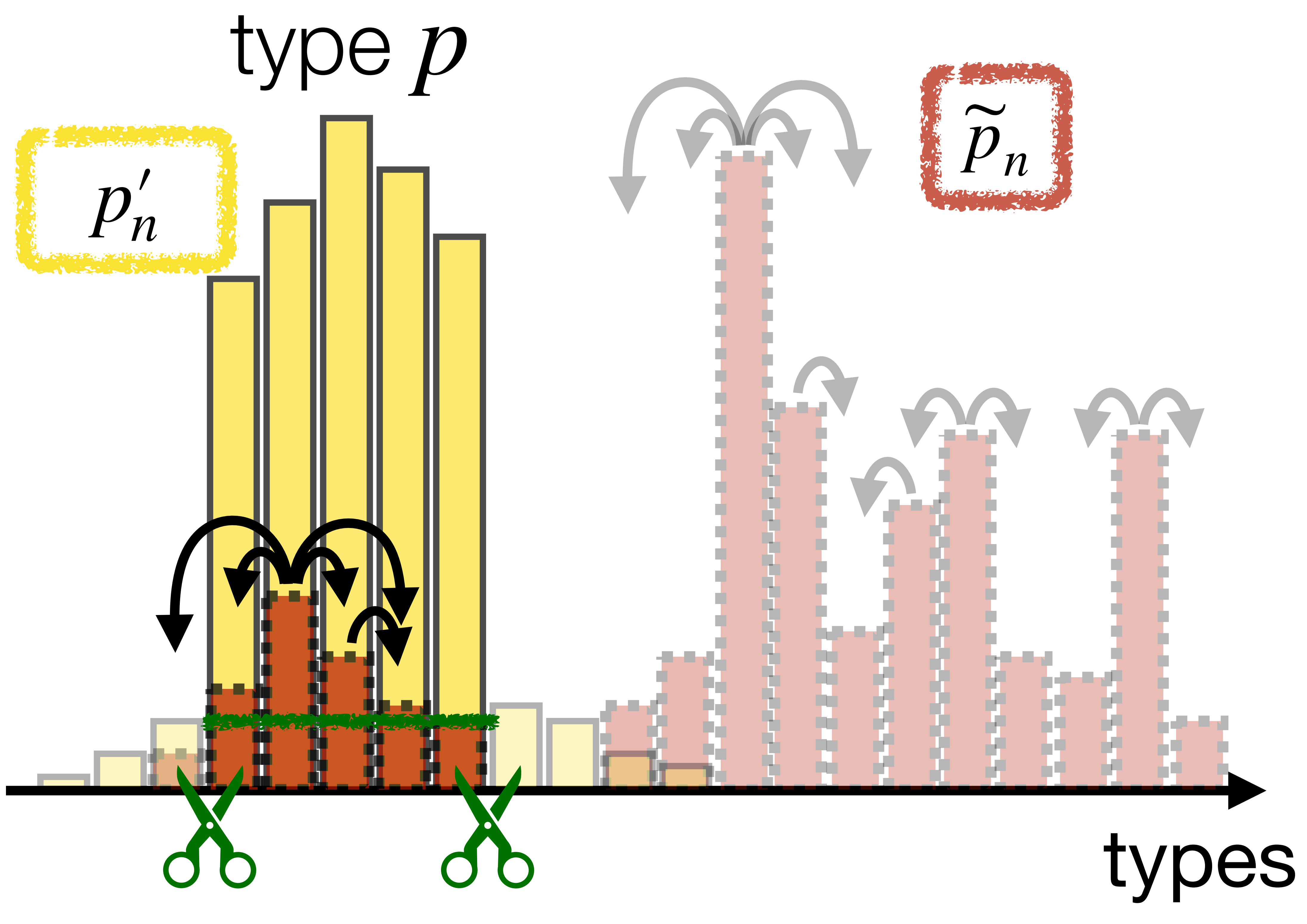
$$p^{\otimes n} \approx_{\eta} p'_n \leq 2^{n\delta'} \widetilde{p}_n$$

η **small**



$$p^{\otimes n} \approx_{\eta} p'_n \leq 2^{n\delta'}\widetilde{p}_n \leq 2^{n(\lambda+\delta')}B_{n,m}(q_n)$$

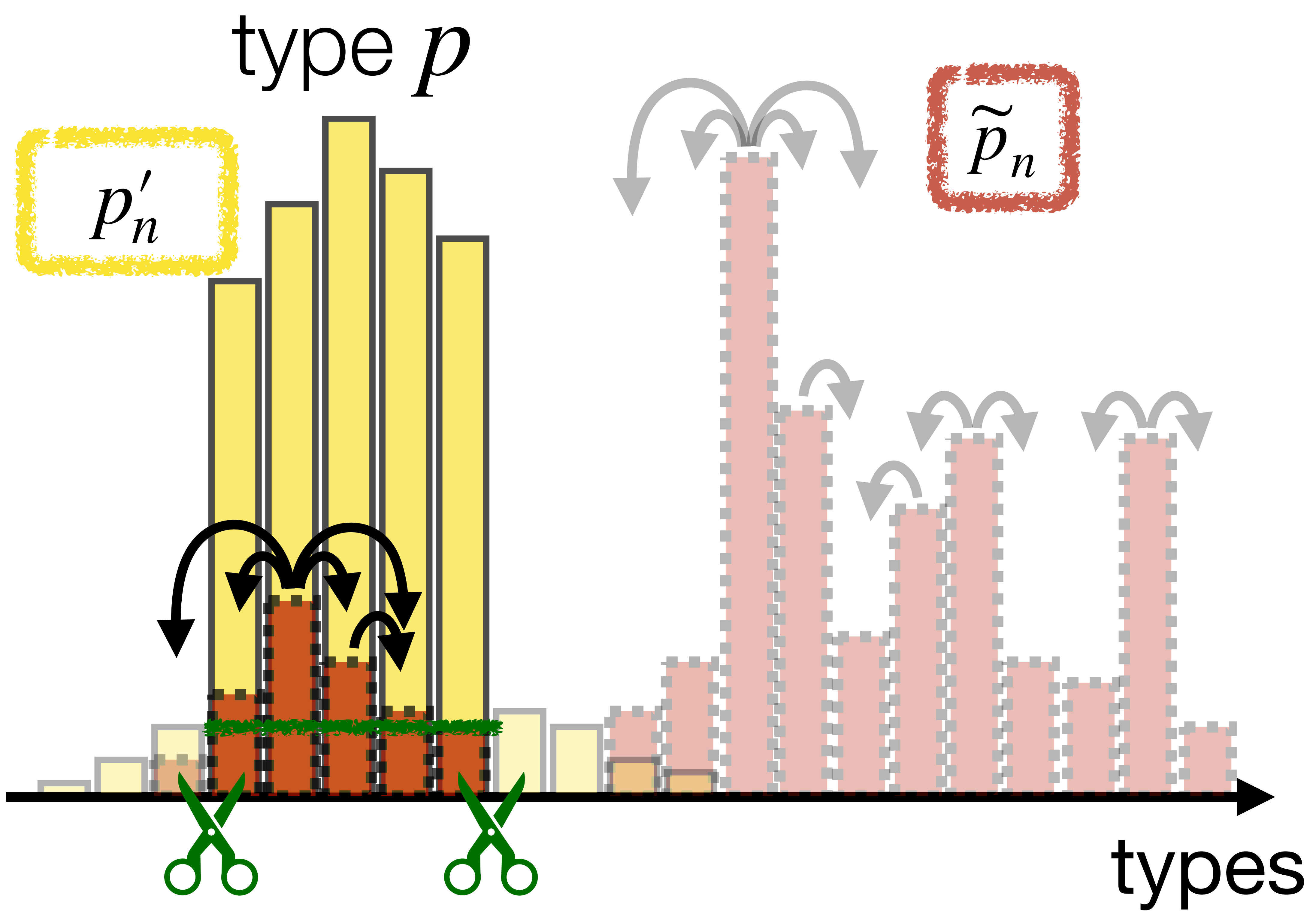
η **small**



$$p^{\otimes n} \approx_{\eta} p'_n \leq 2^{n\delta'} \widetilde{p}_n \leq 2^{n(\lambda+\delta')} B_{n,m}(q_n)$$

η **small**

$$\implies \quad D^{\eta}_{\max}(p^{\otimes n} \,\|\, \mathscr{F}_n) \leq n(\lambda + \delta')$$

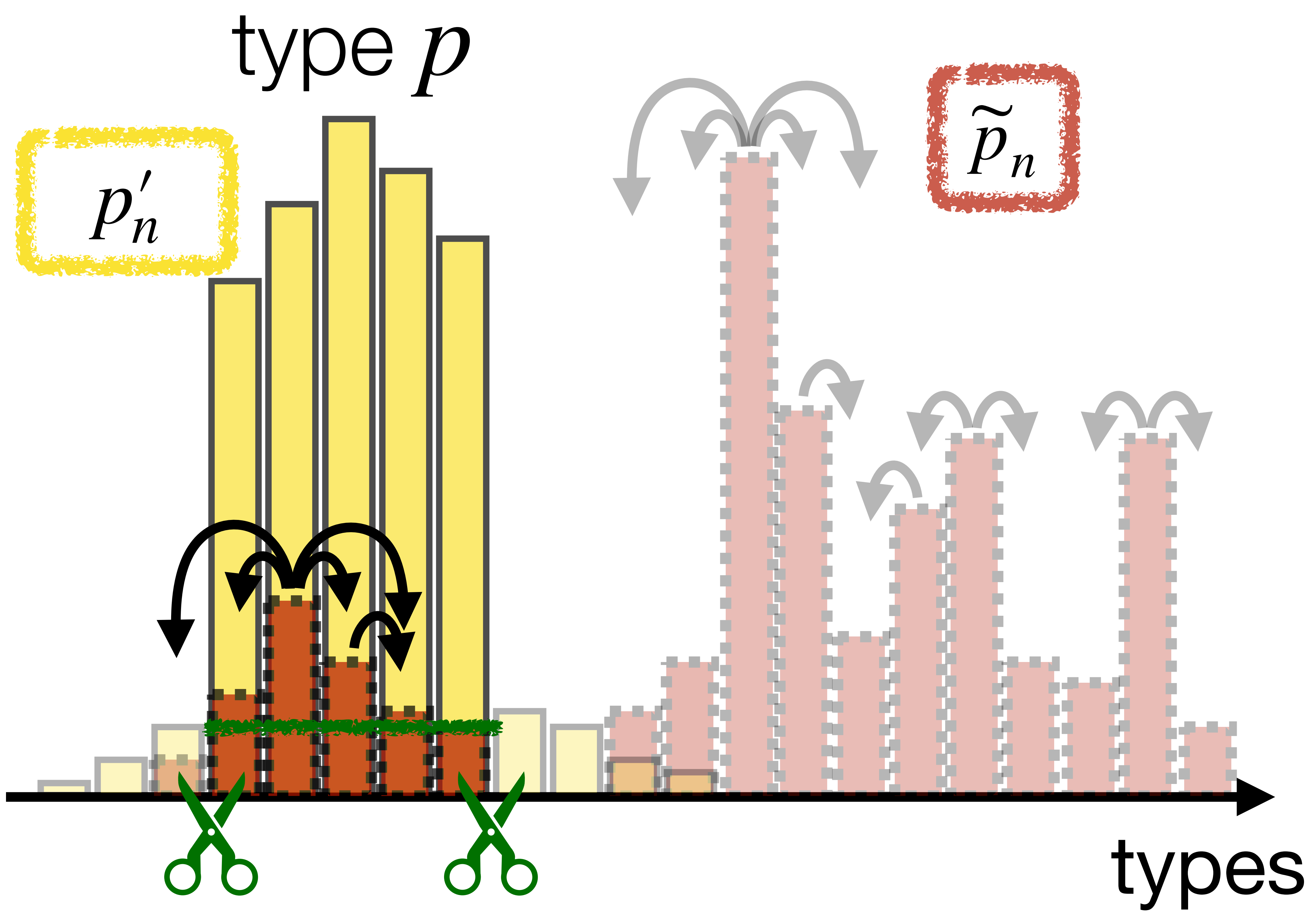


$$p^{\otimes n} \approx_{\eta} p'_n \leq 2^{n\delta'} \widetilde{p}_n \leq 2^{n(\lambda+\delta')} B_{n,m}(q_n)$$

η **small**

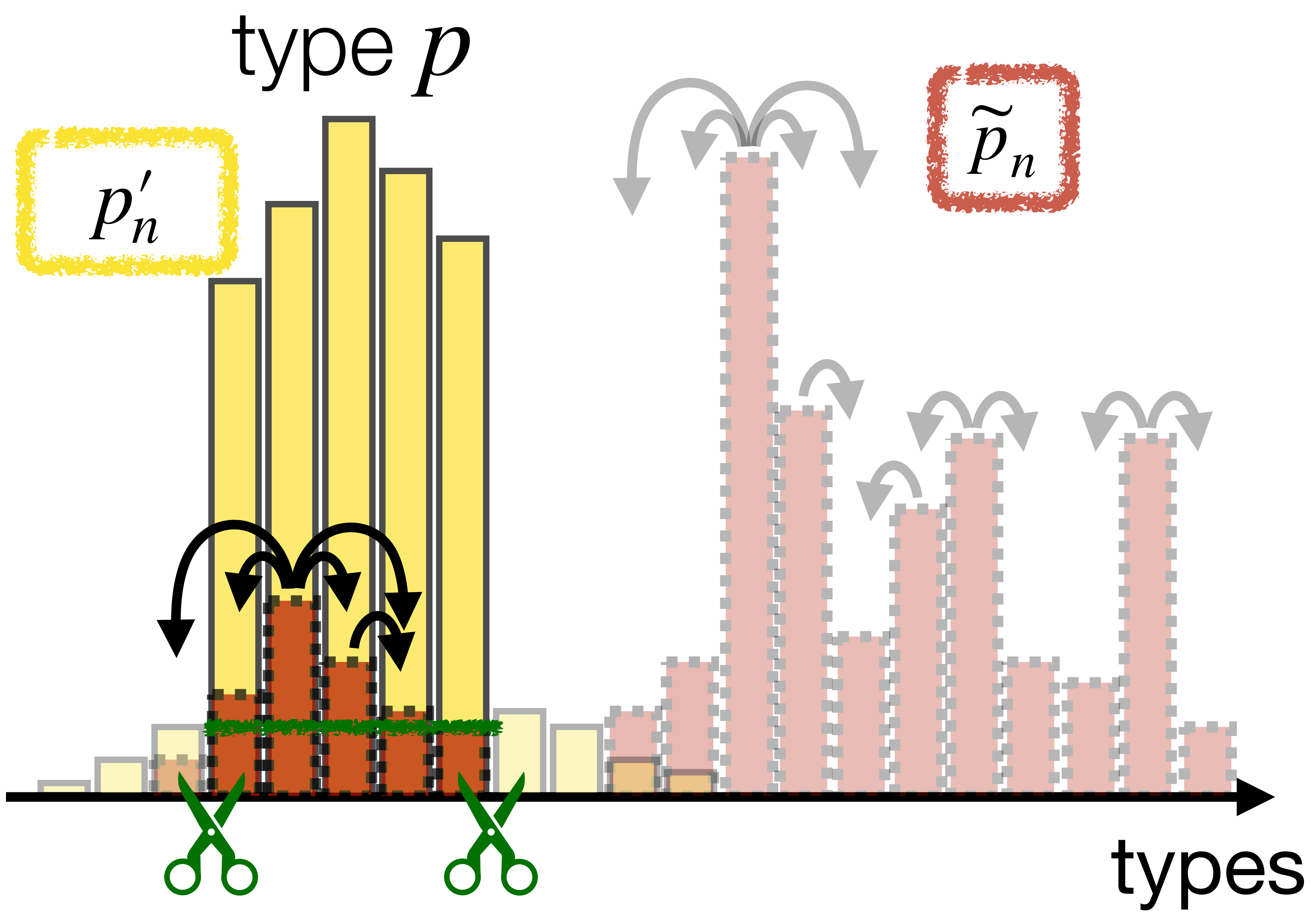
$$\implies \quad D^{\eta}_{\max}(p^{\otimes n} \,\|\, \mathscr{F}_n) \leq n(\lambda + \delta')$$

$$\implies \quad \lim_{\eta \to 0^+} F_p(\eta) \leq \lambda < D^{\infty}(p \| \mathscr{F})$$



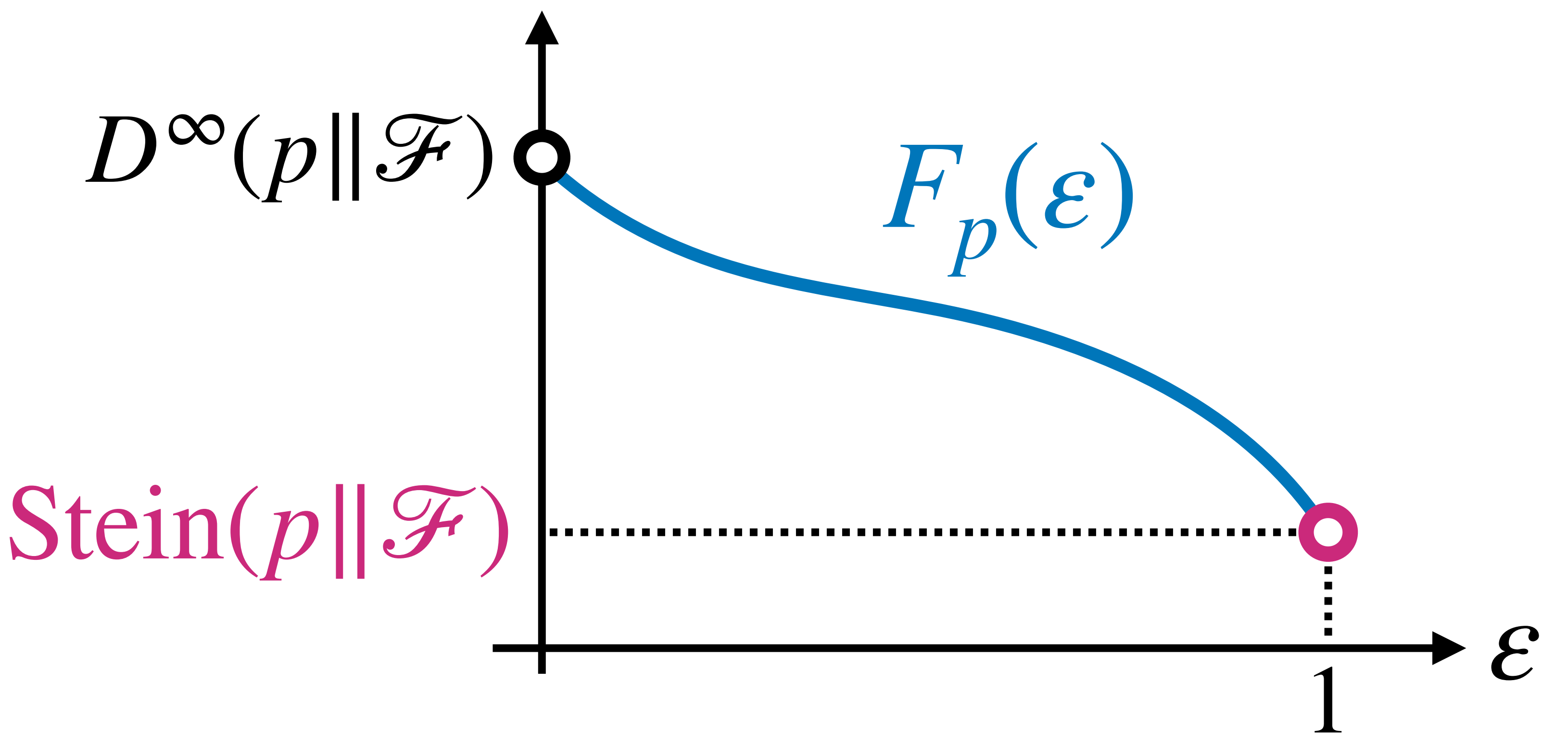
$$p^{\otimes n} \approx_{\eta} p'_n \le 2^{n\delta'} \widetilde{p}_n \le 2^{n(\lambda+\delta')} B_{n,m}(q_n)$$

η **small**



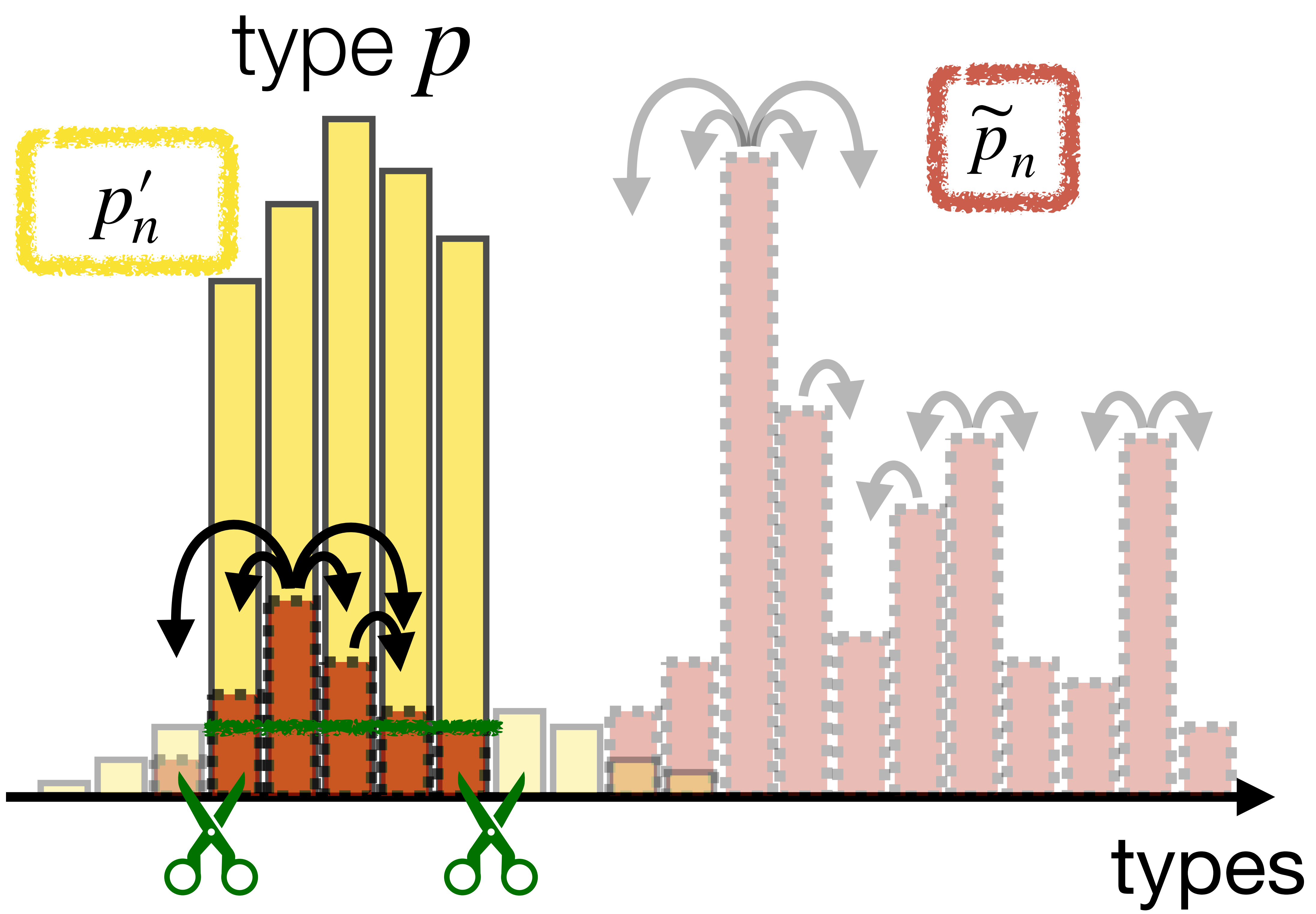
$$\implies \quad D^{\eta}_{\max}(p^{\otimes n} \,\|\, \mathscr{F}_n) \le n(\lambda + \delta')$$

$$\implies \quad D^{\infty}(p\|\mathscr{F}) = \lim_{\eta \to 0^+} F_p(\eta) \le \lambda < D^{\infty}(p\|\mathscr{F})$$



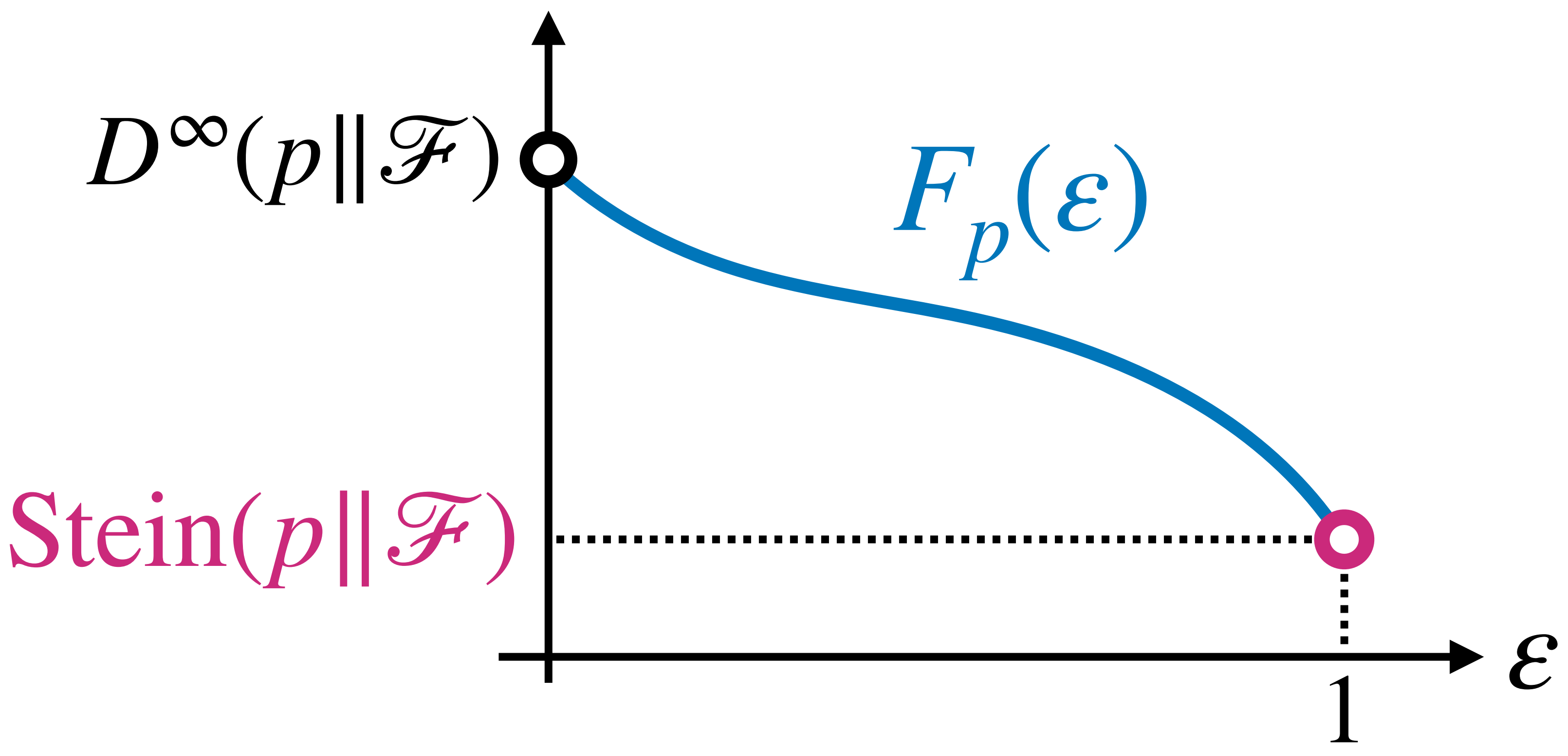
$$p^{\otimes n} \approx_{\eta} p'_n \le 2^{n\delta'} \widetilde{p}_n \le 2^{n(\lambda+\delta')} B_{n,m}(q_n)$$

η small



$$\Longrightarrow \quad D^{\eta}_{\max}(p^{\otimes n} \,\|\, \mathscr{F}_n) \le n(\lambda + \delta')$$

$$\Longrightarrow \quad D^{\infty}(p\|\mathscr{F}) = \lim_{\eta \to 0^+} F_p(\eta) \le \lambda < D^{\infty}(p\|\mathscr{F})$$



CONTRADICTION

End of classical proof

Quantum version

The quantum proof works *morally* in the same way. Q. blurring map $\overline{B}_{n,\delta}$. Then:

Lemma (quantum blurring, informal).

Family $(\mathscr{F}_n)_n$ that obeys the BP axioms. If the sequence of states $(\rho_n)_n$ satisfies

$$\lim_{n\to\infty} \frac{1}{2} \| \rho^{\otimes n} - \rho_n \|_1 = \varepsilon \in (0,1)\,,$$

then

$$\mathrm{Tr}\left(\rho^{\otimes n} - 2^{\delta' n}\overline{B}_{n,\delta}(\rho_n)\right)_+ \underset{n\to\infty}{\longrightarrow} 0\,.$$

$\mathrm{Tr}_+ X :=$ sum of positive eigenvalues of X.

Quantum version

Blurring step is as follows:

Q. blurring map (1.0): $B_{n,m}(\,\cdot\,) := \mathrm{Tr}_m\left[\mathrm{sym}_{n+m}\left((\,\cdot\,) \otimes \sigma_0^{\otimes m}\right)\right]$

Quantum version

Blurring step is as follows:

Q. blurring map (1.0): $$B_{n,\delta}(\,\cdot\,) := \mathrm{Tr}_{\lfloor \delta n \rfloor}\left[\mathrm{sym}_{n+\lfloor \delta n \rfloor}\left((\,\cdot\,) \otimes \sigma_0^{\otimes \lfloor \delta n \rfloor}\right)\right]$$

Quantum version

Blurring step is as follows:

Q. blurring map (1.0): $$B_{n,\delta}(\,\cdot\,) := \mathrm{Tr}_{\lfloor \delta n \rfloor}\left[\mathrm{sym}_{n+\lfloor \delta n \rfloor}\left((\,\cdot\,) \otimes \sigma_0^{\otimes \lfloor \delta n \rfloor}\right)\right]$$

full-support $\sigma_0 \in \mathcal{F}_1$

Quantum version

Blurring step is as follows:

Q. blurring map (1.0): $B_{n,\delta}(\,\cdot\,) := \mathrm{Tr}_{\lfloor \delta n \rfloor}\left[\mathrm{sym}_{n+\lfloor \delta n \rfloor}\left((\,\cdot\,) \otimes \sigma_0^{\otimes \lfloor \delta n \rfloor}\right)\right]$

full-support $\sigma_0 \in \mathscr{F}_1$

$\sigma_0 \geq \lambda_{\min}(\sigma_0)\, I \geq \lambda_{\min}(\sigma_0)\, \rho$

Quantum version

Blurring step is as follows:

Q. blurring map (1.0): $B_{n,\delta}(\,\cdot\,) := \mathrm{Tr}_{\lfloor \delta n \rfloor}\left[\mathrm{sym}_{n+\lfloor \delta n \rfloor}\left((\,\cdot\,) \otimes \sigma_0^{\otimes \lfloor \delta n \rfloor}\right)\right]$

full-support $\sigma_0 \in \mathscr{F}_1$

$\sigma_0 \geq \lambda_{\min}(\sigma_0)\, I \geq \lambda_{\min}(\sigma_0)\, \rho$

Q. blurring map (2.0): $B^{\rho}_{n,\delta}(\,\cdot\,) := \mathrm{Tr}_{\lfloor \delta n \rfloor}\left[\mathrm{sym}_{n+\lfloor \delta n \rfloor}\left((\,\cdot\,) \otimes \rho^{\otimes \lfloor \delta n \rfloor}\right)\right]$

Quantum version

Blurring step is as follows:

Q. blurring map (1.0):

$$B_{n,\delta}(\,\cdot\,) := \mathrm{Tr}_{\lfloor \delta n \rfloor}\left[\mathrm{sym}_{n+\lfloor \delta n \rfloor}\Big((\,\cdot\,) \otimes \sigma_0^{\otimes \lfloor \delta n \rfloor}\Big)\right]$$

full-support $\sigma_0 \in \mathscr{F}_1$

$$\sigma_0 \geq \lambda_{\min}(\sigma_0)\, I \geq \lambda_{\min}(\sigma_0)\, \rho$$

Q. blurring map (2.0):

$$B^{\rho}_{n,\delta}(\,\cdot\,) := \mathrm{Tr}_{\lfloor \delta n \rfloor}\left[\mathrm{sym}_{n+\lfloor \delta n \rfloor}\Big((\,\cdot\,) \otimes \rho^{\otimes \lfloor \delta n \rfloor}\Big)\right]$$

To prove:

$$\lim_{n\to\infty} \frac{1}{2}\, \|\, \rho^{\otimes n} - \rho_n \,\|_1 = \varepsilon \in (0,1) \qquad \Longrightarrow \qquad \mathrm{Tr}\,\Big(\rho^{\otimes n} - 2^{\delta' n} B^{\rho}_{n,\delta}(\rho_n)\Big)_+ \underset{n\to\infty}{\longrightarrow} 0\,.$$

Step 1. Wlog, assume ρ_n permutationally symmetric.

Step 1. Wlog, assume ρ_n permutationally symmetric.

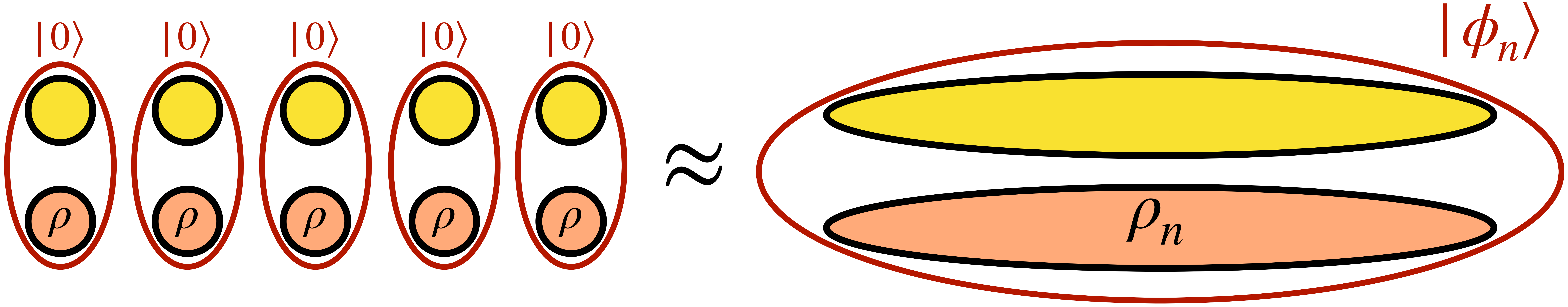
Step 2. Purification:

$\rho_n \approx_\varepsilon \rho^{\otimes n} \Rightarrow \exists$ perm. symm. purifications $|\phi_n\rangle$, $|0^{\otimes n}\rangle$ s.t. $\langle\phi_n|0^{\otimes n}\rangle \geq 1 - \varepsilon$.

Step 1. Wlog, assume ρ_n permutationally symmetric.

Step 2. Purification:

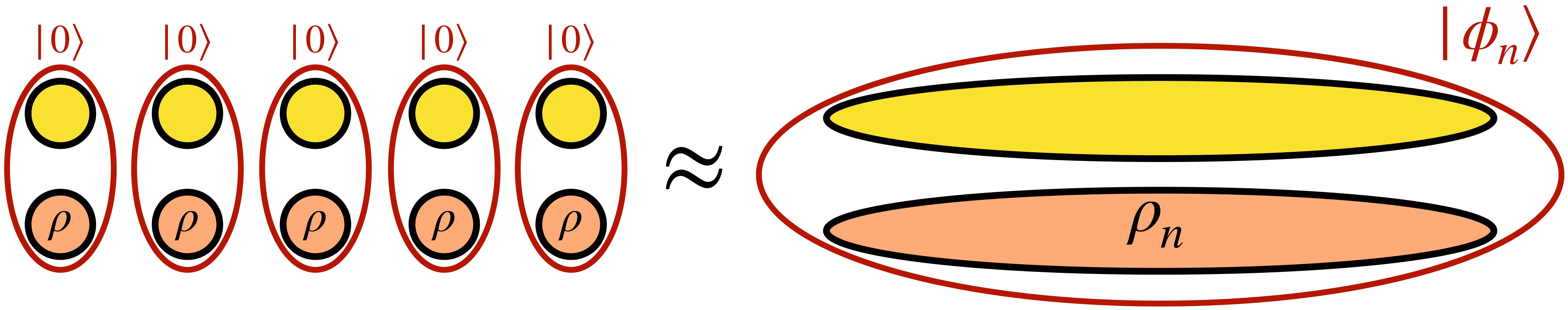
$\rho_n \approx_\varepsilon \rho^{\otimes n} \Rightarrow \exists$ perm. symm. purifications $|\phi_n\rangle$, $|0^{\otimes n}\rangle$ s.t. $\langle \phi_n | 0^{\otimes n}\rangle \geq 1 - \varepsilon$.



Step 1. Wlog, assume ρ_n permutationally symmetric.

Step 2. Purification:

$\rho_n \approx_\varepsilon \rho^{\otimes n} \Rightarrow \exists$ perm. symm. purifications $|\phi_n\rangle$, $|0^{\otimes n}\rangle$ s.t. $\langle \phi_n | 0^{\otimes n}\rangle \geq 1 - \varepsilon$.



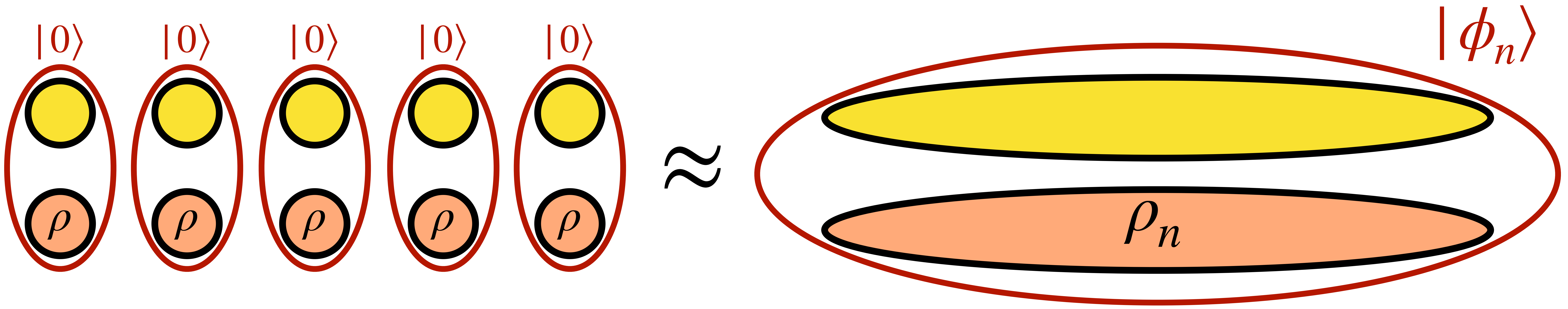
Now, by data processing:

$$\mathrm{Tr}\left(\rho^{\otimes n} - 2^{\delta' n} B^{\rho}_{n,\delta}(\rho_n)\right)_+ \leq \mathrm{Tr}\left(|0\rangle\langle 0|^{\otimes n} - 2^{\delta' n} B^{|0\rangle\langle 0|}_{n,\delta}(\phi_n)\right)_+$$

Step 1. Wlog, assume ρ_n permutationally symmetric.

Step 2. Purification:

$\rho_n \approx_\varepsilon \rho^{\otimes n} \Rightarrow \exists$ perm. symm. purifications $|\phi_n\rangle$, $|0^{\otimes n}\rangle$ s.t. $\langle \phi_n | 0^{\otimes n}\rangle \geq 1 - \varepsilon$.



Now, by data processing:

$$\mathrm{Tr}\left(\rho^{\otimes n} - 2^{\delta' n} B^{\rho}_{n,\delta}(\rho_n)\right)_+ \leq \mathrm{Tr}\left(|0\rangle\langle 0|^{\otimes n} - 2^{\delta' n} B^{|0\rangle\langle 0|}_{n,\delta}(\phi_n)\right)_+$$

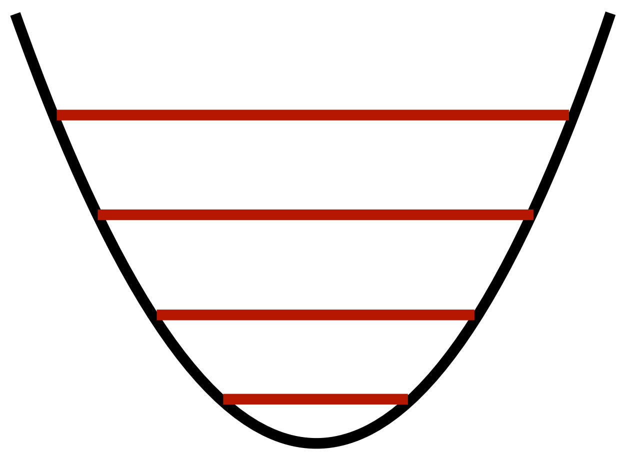
$$= \mathrm{Tr}\left(|0\rangle\langle 0|^{\otimes n} - 2^{\delta' n}\, \Pi_n^{\mathrm{Sym}} B^{|0\rangle\langle 0|}_{n,\delta}(\phi_n)\, \Pi_n^{\mathrm{Sym}}\right)_+ .$$

Step 3. Bosonic lifting $\longrightarrow$ embed everything into Fock space!

Step 3. **Bosonic lifting** $\longrightarrow$ embed everything into Fock space!

$$\mathrm{Sym}^n(\mathbb{C}^d) \subseteq L^2(\mathbb{R}^{d-1})$$

$$|0\rangle^{\otimes n} \longmapsto |\mathrm{vac}\rangle$$



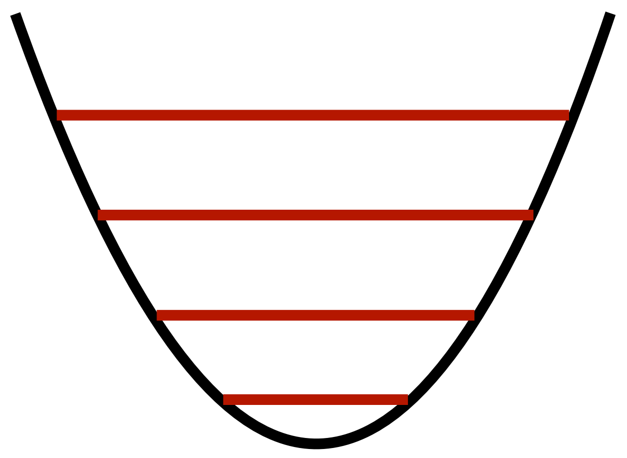
$\otimes\,(d-1)$

Step 3. Bosonic lifting $\longrightarrow$ embed everything into Fock space!

$d = 2:$

$$\mathrm{Sym}^n(\mathbb{C}^2) \subseteq L^2(\mathbb{R})$$

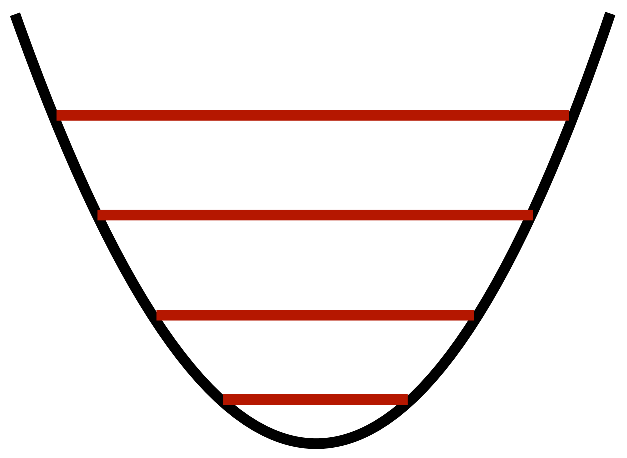
$$|0\rangle^{\otimes n} \longmapsto |\mathrm{vac}\rangle$$



Step 3. **Bosonic lifting** $\longrightarrow$ embed everything into Fock space!

$d = 2:$

$$\mathrm{Sym}^n(\mathbb{C}^2) \subseteq L^2(\mathbb{R})$$



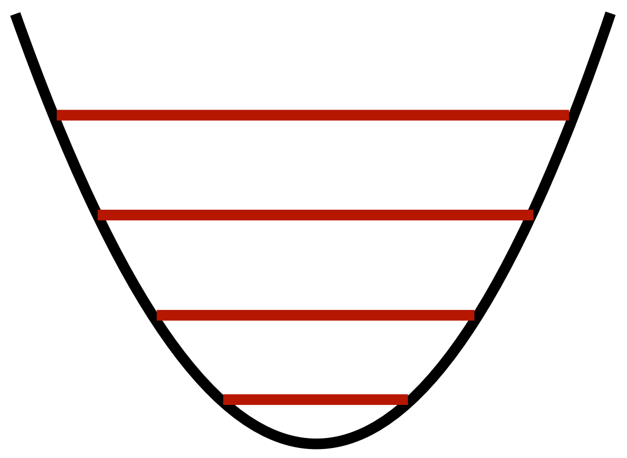
$$|0\rangle^{\otimes n} \longmapsto |\mathrm{vac}\rangle$$

$$\frac{1}{\sqrt{n}}\left(|0...01\rangle + \ldots + |10...0\rangle\right) \longmapsto |1\rangle$$

$$\vdots$$

Step 3. **Bosonic lifting** $\longrightarrow$ embed everything into Fock space!

$$d = 2: \qquad \mathrm{Sym}^n(\mathbb{C}^2) \subseteq L^2(\mathbb{R})$$



$$|0\rangle^{\otimes n} \longmapsto |\mathrm{vac}\rangle$$

$$\frac{1}{\sqrt{n}}\left(|0...01\rangle + \ldots + |10...0\rangle\right) \longmapsto |1\rangle$$

$$\vdots$$

Action of blurring for large n: *any* $|1\rangle$ *in the input has a fixed probability of ending up in* $\lfloor \delta n \rfloor / n$ *systems that get traced away.*

Step 3. **Bosonic lifting** $\longrightarrow$ embed everything into Fock space!

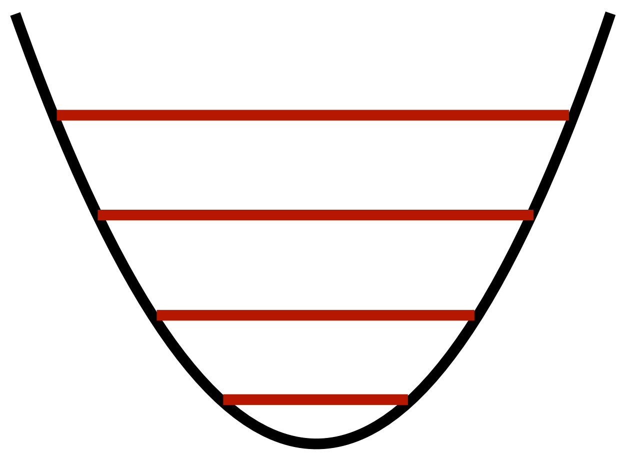
$d = 2:$

$$\mathrm{Sym}^n(\mathbb{C}^2) \subseteq L^2(\mathbb{R})$$

$$|0\rangle^{\otimes n} \longmapsto |\mathrm{vac}\rangle$$

$$\frac{1}{\sqrt{n}}\left(|0...01\rangle + \ldots + |10...0\rangle\right) \longmapsto |1\rangle$$

$$\vdots$$

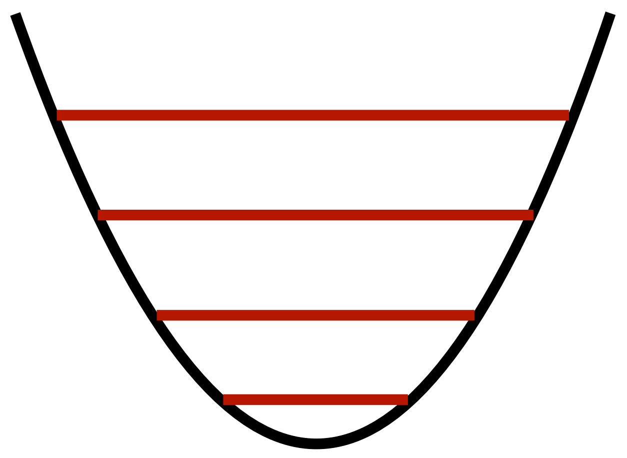


Action of blurring for large n: *any* $|1\rangle$ *in the input has a fixed probability of ending up in* $\lfloor \delta n \rfloor / n$ *systems that get traced away.*

What channel scatters away excitations on a bosonic mode?

Step 3. **Bosonic lifting** $\longrightarrow$ embed everything into Fock space!

$$d = 2: \qquad \mathrm{Sym}^n(\mathbb{C}^2) \subseteq L^2(\mathbb{R})$$



$$|0\rangle^{\otimes n} \longmapsto |\mathrm{vac}\rangle$$

$$\frac{1}{\sqrt{n}}\left(|0...01\rangle + \ldots + |10...0\rangle\right) \longmapsto |1\rangle$$

$$\vdots$$

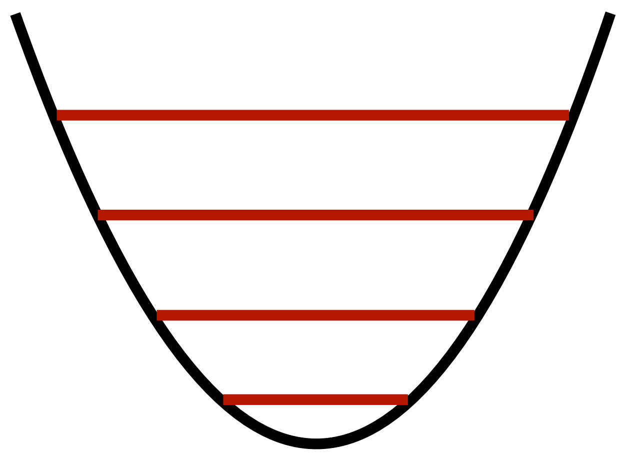
Action of blurring for large *n*: *any* $|1\rangle$ *in the input has a fixed probability of ending up in* $\lfloor \delta n \rfloor / n$ *systems that get traced away.*

What channel scatters away excitations on a bosonic mode?

A **pure loss** channel!

Step 3. **Bosonic lifting** $\longrightarrow$ embed everything into Fock space!

$$d = 2: \qquad \mathrm{Sym}^n(\mathbb{C}^2) \subseteq L^2(\mathbb{R})$$



$$|0\rangle^{\otimes n} \longmapsto |\mathrm{vac}\rangle$$

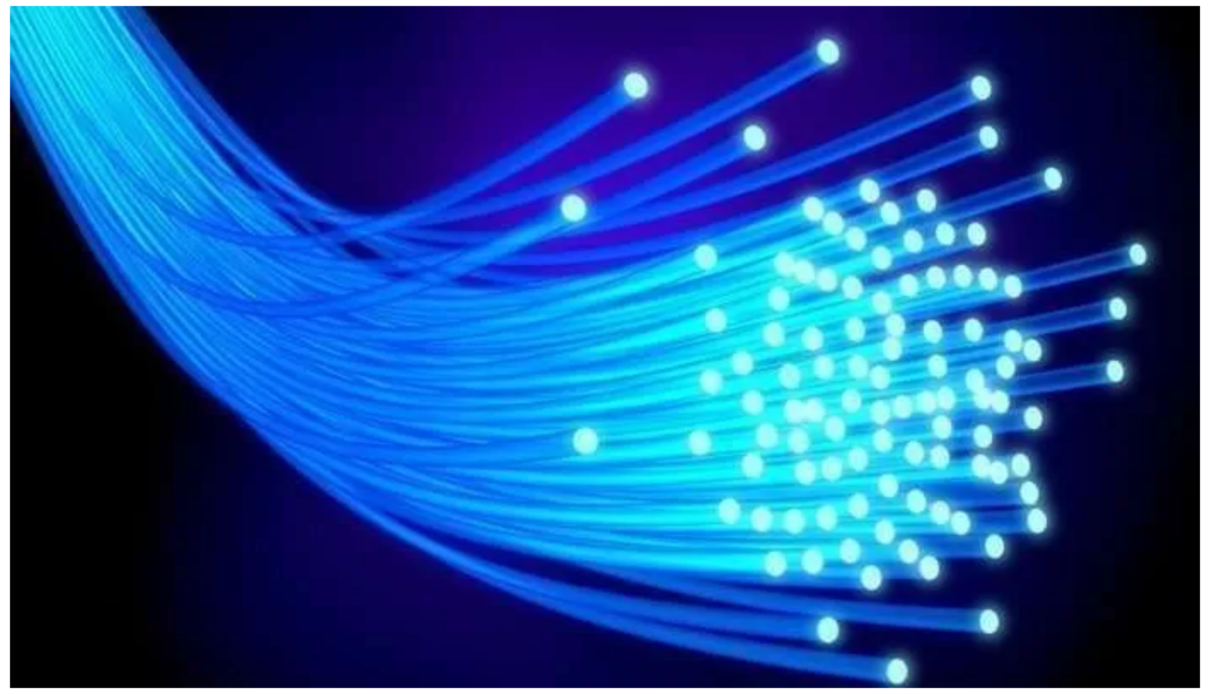
$$\frac{1}{\sqrt{n}}\left(|0...01\rangle + \ldots + |10...0\rangle\right) \longmapsto |1\rangle$$

$$\vdots$$

Action of blurring for large n: *any* $|1\rangle$ *in the input has a fixed probability of ending up in* $\lfloor \delta n \rfloor / n$ *systems that get traced away.*

What channel scatters away excitations on a bosonic mode?

A **pure loss** channel!



Step 3. Bosonic lifting $\longrightarrow$ embed everything into Fock space!

Pure loss channel:

$$\mathcal{E}_\lambda(\,\cdot\,) := \mathrm{Tr}_2\, U_\lambda \big((\,\cdot\,) \otimes |0\rangle\langle 0| \big)\, U_\lambda^\dagger$$

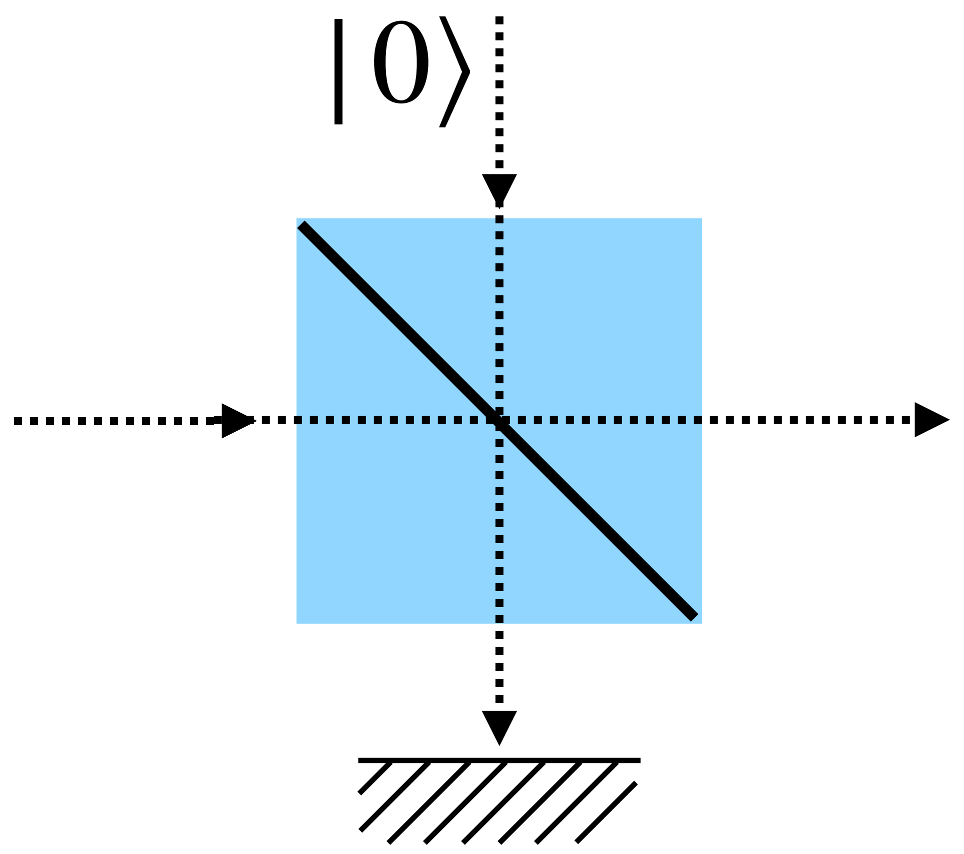
Step 3. **Bosonic lifting** $\longrightarrow$ embed everything into Fock space!

Pure loss channel:

beam splitter unitary

$$\mathscr{E}_\lambda(\,\cdot\,) := \mathrm{Tr}_2\, U_\lambda \big((\,\cdot\,) \otimes |0\rangle\langle 0| \big)\, U_\lambda^\dagger$$

$$U_\lambda := e^{\arccos\sqrt{\lambda}\,(a^\dagger b - a b^\dagger)}$$



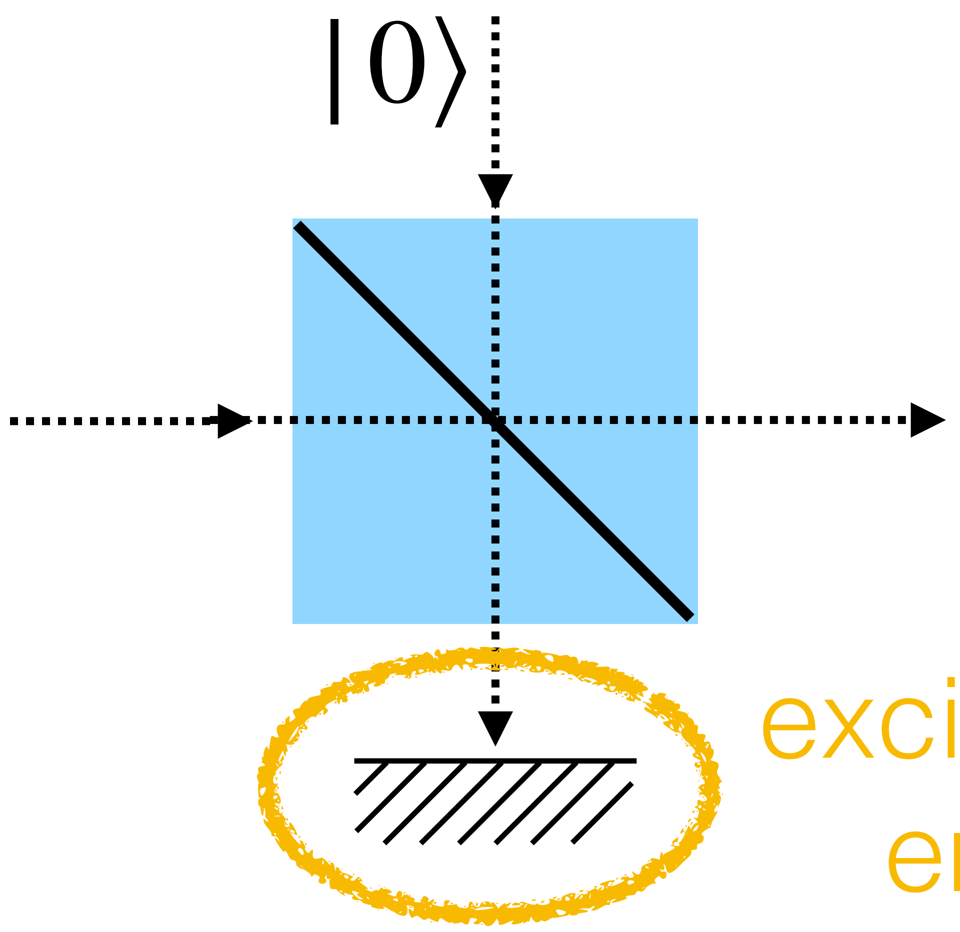
Step 3. **Bosonic lifting** $\longrightarrow$ embed everything into Fock space!

Pure loss channel:

beam splitter unitary

$$\mathscr{E}_\lambda(\,\cdot\,) := \mathrm{Tr}_2\, U_\lambda \big((\,\cdot\,) \otimes |0\rangle\!\langle 0| \big)\, U_\lambda^\dagger$$

$$U_\lambda := e^{\arccos\sqrt{\lambda}\,(a^\dagger b - a b^\dagger)}$$



excitations lost to environment

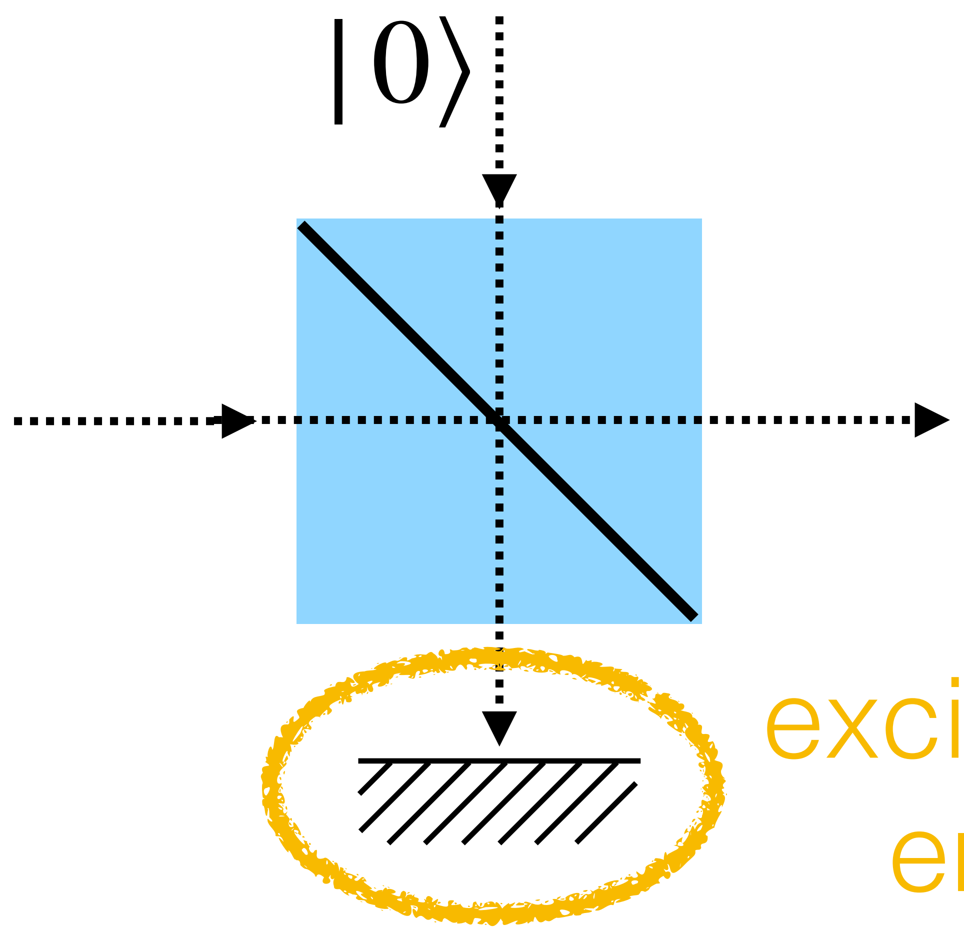
Step 3. **Bosonic lifting** $\longrightarrow$ embed everything into Fock space!

Pure loss channel:

beam splitter unitary

$$\mathscr{E}_\lambda(\,\cdot\,) := \mathrm{Tr}_2\, U_\lambda \big((\,\cdot\,) \otimes |0\rangle\langle 0|\big)\, U_\lambda^\dagger$$

$$U_\lambda := e^{\arccos\sqrt{\lambda}\,(a^\dagger b - a b^\dagger)}$$



Formal statement of bosonic lifting ($d = 2$):

$$\Pi_n^{\mathrm{Sym}}\, B_{n,\delta}^{|0\rangle\langle 0|}(\,\cdot\,)\, \Pi_n^{\mathrm{Sym}} \xrightarrow[n \to \infty]{s} \mathscr{E}_{\lambda(\delta)} \circ \mathscr{D}_{\mu(\delta)}$$

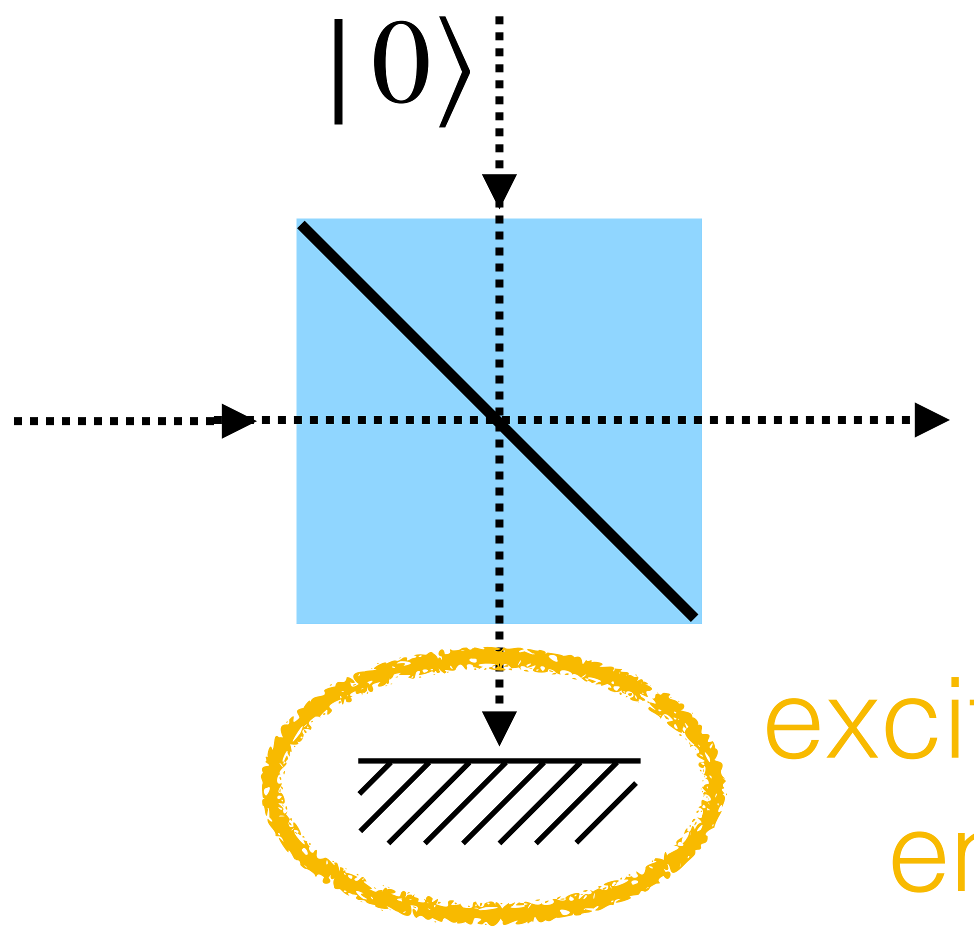
Step 3. **Bosonic lifting** $\longrightarrow$ embed everything into Fock space!

Pure loss channel:

beam splitter unitary

$$\mathscr{E}_\lambda(\,\cdot\,) := \mathrm{Tr}_2\, U_\lambda \big((\,\cdot\,) \otimes |0\rangle\langle 0| \big)\, U_\lambda^\dagger$$

$$U_\lambda := e^{\arccos\sqrt{\lambda}\,(a^\dagger b - a b^\dagger)}$$



Formal statement of bosonic lifting ($d = 2$):

strong convergence

$$\Pi_n^{\mathrm{Sym}}\, B_{n,\delta}^{|0\rangle\langle 0|}(\,\cdot\,)\, \Pi_n^{\mathrm{Sym}} \xrightarrow[n\to\infty]{s} \mathscr{E}_{\lambda(\delta)} \circ \mathscr{D}_{\mu(\delta)}$$

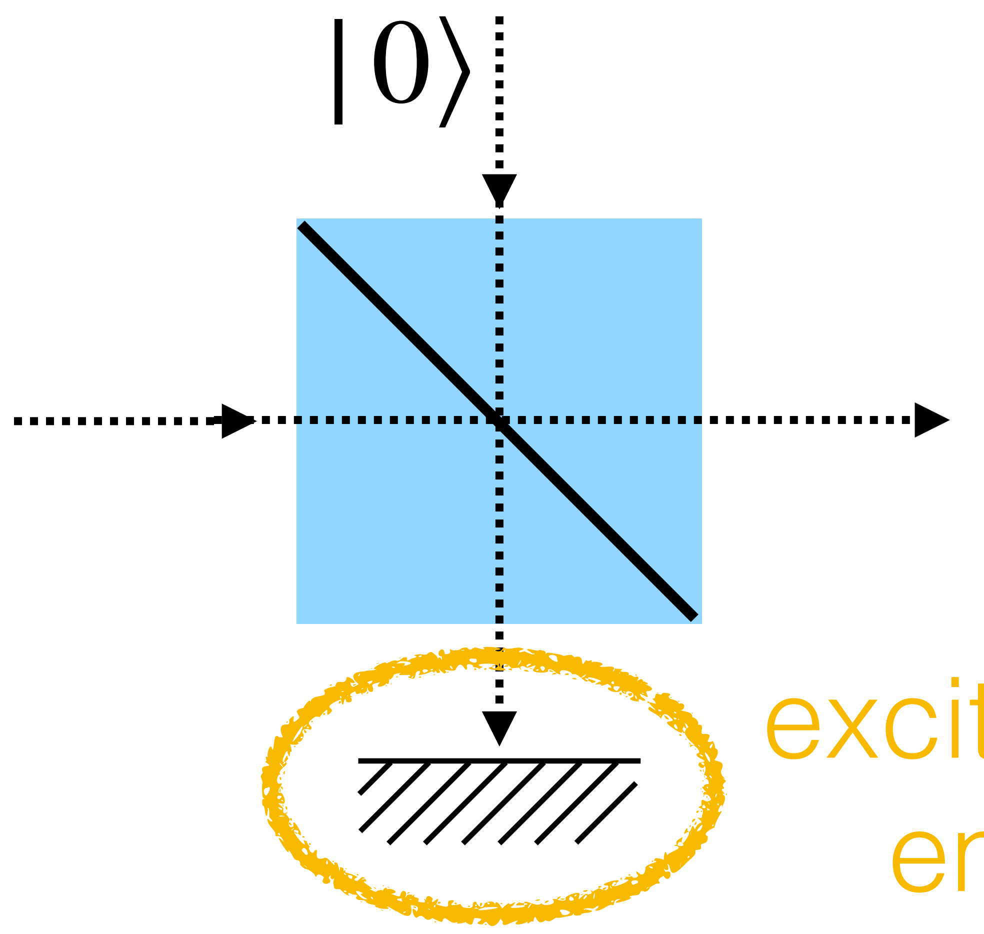
Step 3. **Bosonic lifting** $\longrightarrow$ embed everything into Fock space!

Pure loss channel:

beam splitter unitary

$$\mathcal{E}_\lambda(\,\cdot\,) := \mathrm{Tr}_2\, U_\lambda \big((\,\cdot\,) \otimes |0\rangle\!\langle 0|\big)\, U_\lambda^\dagger$$

$$U_\lambda := e^{\arccos\sqrt{\lambda}\,(a^\dagger b - a b^\dagger)}$$



Formal statement of bosonic lifting ($d = 2$):

strong convergence

$$\Pi_n^{\mathrm{Sym}}\, B_{n,\delta}^{|0\rangle\!\langle 0|}(\,\cdot\,)\, \Pi_n^{\mathrm{Sym}} \xrightarrow[n\to\infty]{s} \mathcal{E}_{\lambda(\delta)} \circ \mathcal{D}_{\mu(\delta)}$$

$$\mathcal{D}_\mu(\,\cdot\,) := \mu^{a^\dagger a}(\,\cdot\,)\mu^{a^\dagger a}, \qquad \lambda(\delta) := \frac{1}{1+\delta(1+\delta)}, \quad \mu(\delta) := \frac{\sqrt{1+\delta(1+\delta)}}{1+\delta}.$$

Step 4. Solve the problem in Fock space:

Step 4. Solve the problem in Fock space:

$$\mathrm{Tr}\left(|0\rangle\langle 0| - M\mathcal{E}_\lambda(\phi)\right)_+ \xrightarrow[M\to\infty]{?} 0 \qquad \forall\ |\phi\rangle$$

Step 4. Solve the problem in Fock space:

$$\mathrm{Tr}\left(|0\rangle\langle 0| - M\mathcal{E}_\lambda(\phi)\right)_+ \xrightarrow[M\to\infty]{?} 0 \qquad \forall\ |\phi\rangle$$

Counterexample: coherent states! $|\alpha\rangle := e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$:

$$\mathcal{E}_\lambda(|\alpha\rangle\langle\alpha|) = |\sqrt{\lambda}\alpha\rangle\langle\sqrt{\lambda}\alpha|$$

Step 4. Solve the problem in Fock space:

$$\mathrm{Tr}\left(|0\rangle\langle 0| - M\mathcal{E}_\lambda(\phi)\right)_+ \xrightarrow[M\to\infty]{?} 0 \qquad \forall\ |\phi\rangle$$

Counterexample: coherent states! $|\alpha\rangle := e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$:

$$\mathcal{E}_\lambda(|\alpha\rangle\langle\alpha|) = |\sqrt{\lambda}\alpha\rangle\langle\sqrt{\lambda}\alpha|$$

Fix: randomise δ (i.e. randomise λ). Then it is true that

$$\mathrm{Tr}\left(|0\rangle\langle 0| - M\int_0^\lambda \frac{d\lambda'}{\lambda}\,\mathcal{E}_{\lambda'}(\phi)\right)_+ \xrightarrow[M\to\infty]{} 0 \qquad \forall\ |\phi\rangle$$

Step 4. Solve the problem in Fock space:

$$\mathrm{Tr}\left(|0\rangle\langle 0| - M\mathcal{E}_\lambda(\phi)\right)_+ \xrightarrow[M\to\infty]{?} 0 \qquad \forall\ |\phi\rangle$$

Counterexample: coherent states! $|\alpha\rangle := e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$:

$$\mathcal{E}_\lambda(|\alpha\rangle\langle\alpha|) = |\sqrt{\lambda}\alpha\rangle\langle\sqrt{\lambda}\alpha|$$

Fix: randomise δ (i.e. randomise λ). Then it is true that

$$\mathrm{Tr}\left(|0\rangle\langle 0| - M\int_0^\lambda \frac{d\lambda'}{\lambda}\mathcal{E}_{\lambda'}(\phi)\right)_+ \xrightarrow[M\to\infty]{} 0 \qquad \forall\ |\phi\rangle \qquad ✅$$

Summary

- Generalised quantum Stein's lemma:

Stein exponent for resource testing = regularised relative entropy of resource

L, *IEEE Trans. Inf. Theory* 2025 (arXiv:2408.06410)

Summary

- Generalised quantum Stein's lemma:

Stein exponent for resource testing = regularised relative entropy of resource

⇝ asymptotic reversibiliy of q. resources

L, *IEEE Trans. Inf. Theory* 2025 (arXiv:2408.06410)

Summary

- Generalised quantum Stein's lemma:

Stein exponent for resource testing = regularised relative entropy of resource

⇝ asymptotic reversibiliy of q. resources

- Two techniques: BLURRING / Bosonic lifting

L, *IEEE Trans. Inf. Theory* 2025 (arXiv:2408.06410)

Summary

- Generalised quantum Stein's lemma:

Stein exponent for resource testing = regularised relative entropy of resource

$\rightsquigarrow$ asymptotic reversibiliy of q. resources

- Two techniques: BLURRING / Bosonic lifting

⭐ I'm hiring: ludovico.lami@gmail.com / X / Quantiki

L, *IEEE Trans. Inf. Theory* 2025 (arXiv:2408.06410)

Summary

- Generalised quantum Stein's lemma:

Stein exponent for resource testing = regularised relative entropy of resource

⟿ asymptotic reversibiliy of q. resources

- Two techniques:

BLURRING

Bosonic lifting

⭐ I'm hiring: ludovico.lami@gmail.com / X / Quantiki

Thank you!

L, *IEEE Trans. Inf. Theory* 2025 (arXiv:2408.06410)