Bridging magic and non-Gaussian resources via Gottesman-Kitaev-Preskill encoding

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PRX Quantum **6**, 010330

Introduction

Quantum Resources for quantum advantages

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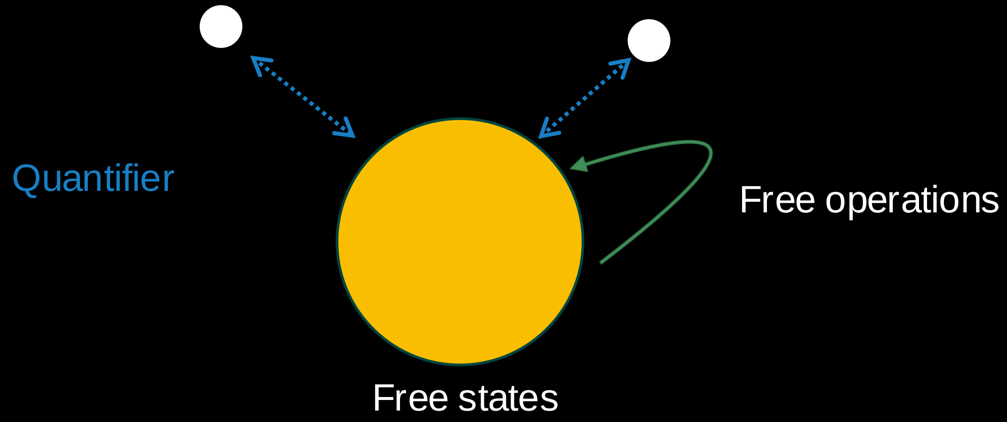
- **Big goal**: Quantitative understanding of quantum resources enabling quantum advantages underlying given physical and operational settings.

- **Quantum resource theories**: Framework to deal with **quantification** and **manipulation** of quantum resources

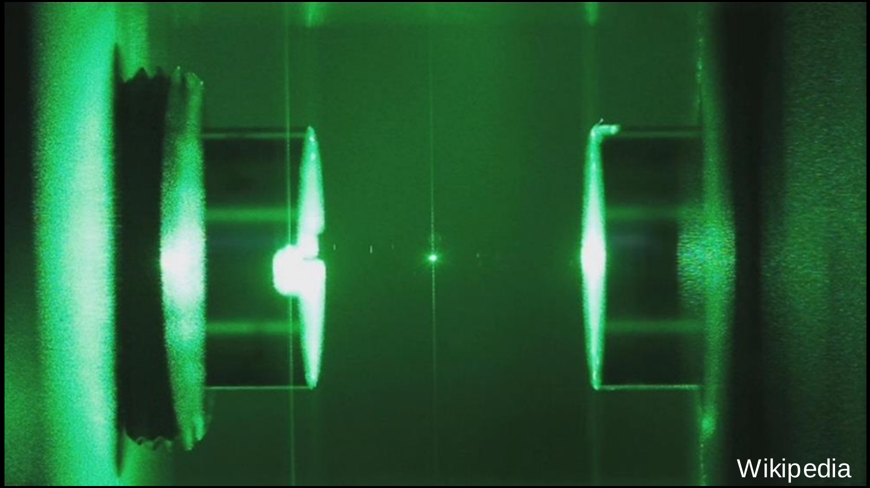


- Here, we study quantum computational resources relevant in CV and DV systems.

Continuous Variables

- Quantum information encoded in q. modes
  - Harmonic oscillators
- Relevant observables $(\hat{q}, \hat{p})$ have continuous spectrum
  - Infinite dimensional Hilbert space

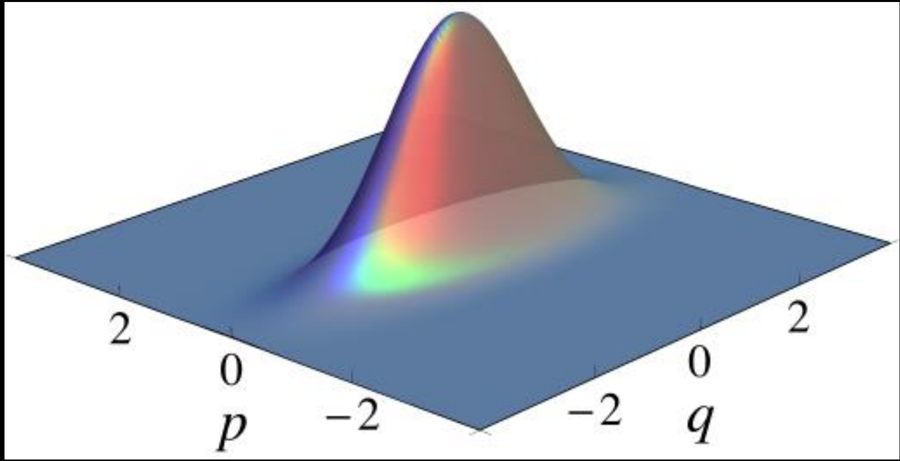
$$[\hat{q}, \hat{p}] = i$$



Wikipedia

Gaussian quantum optics

- Gaussian quantum optics offers rich playground
- Important since it can be implemented experimentally
- Many analytical tools



Gaussian quantum optics

- Gaussian unitary operations $U = e^{iH}$

$$H = \frac{1}{2}\boldsymbol{r}^T \mathrm{H} \boldsymbol{r} + \bar{\boldsymbol{r}}\boldsymbol{r}$$

Displacements

Symplectic

- Gaussian states $\rho_G = \dfrac{e^{-\beta H}}{\mathrm{Tr}[e^{-\beta H}]}$

Gaussian states are fully characterized by covariance matrix and mean!

Gaussian quantum optics

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- Displacement operators (CV Paulis): $\hat{D}(\boldsymbol{r}) = \prod_{j=1}^{n} e^{-ir_{p_j} r_{q_j}/2} e^{-ir_{q_j}\hat{p}_j} e^{ir_{p_j}\hat{q}_j}$
- Symplectic unitaries: $\hat{U}_G \hat{D}(\boldsymbol{r}) \hat{U}_G^\dagger = \hat{D}(S\boldsymbol{r})$

$$S\Omega S^T = \Omega$$

$$\Omega = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}$$

Gaussian quantum optics

Simulatability

Any quantum process that begins with

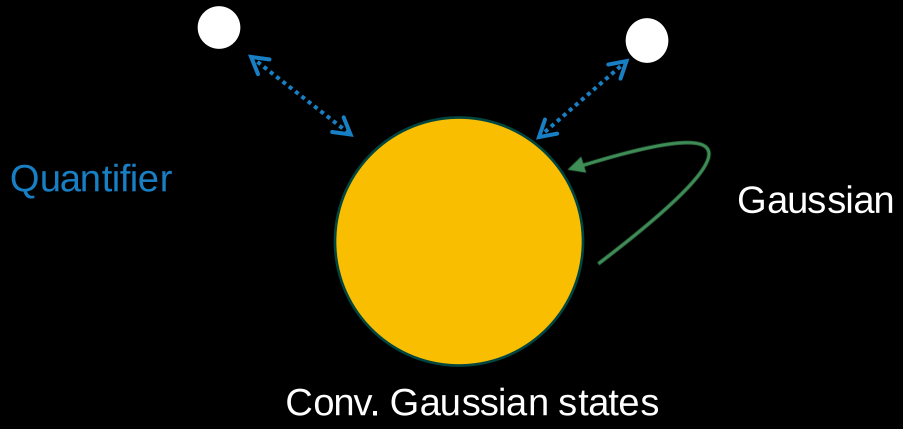
- Gaussian states
- Performs only Gaussian unitaries
- Involves only measurements of canonical operators (including finite losses)

can be simulated efficiently on a classical computer.

[Mari, Eisert, Phys. Rev. Lett. ’12]

Non-Gaussianity

- Gaussian circuits are classical simulatable
- Non-Gaussianity is a necessary resource for quantum advantage



[Takagi, Zhuang, Phys. Rev. A ’18]

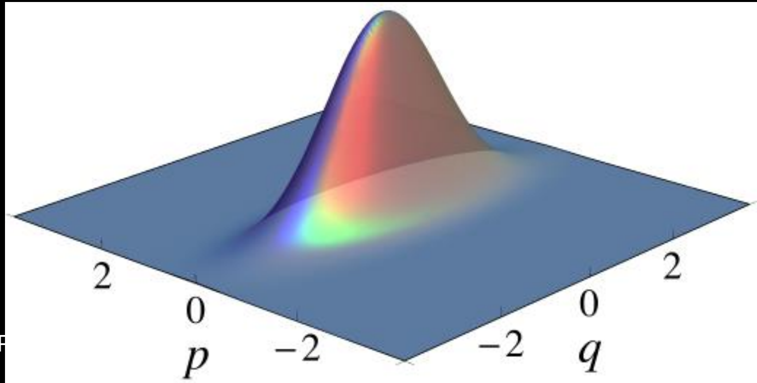
[Albarelli et al., Phys. Rev. A ’18]

CV Wigner function

- Phase-space representation of a quantum state
  - Fully equivalent to the density operator formalism

$$W_{\hat{\rho}}(\boldsymbol{r}) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{x} e^{i\boldsymbol{r_p}\boldsymbol{x}} \left\langle \boldsymbol{r_q} + \frac{\boldsymbol{x}}{2} \middle| \hat{\rho} \middle| \boldsymbol{r_q} - \frac{\boldsymbol{x}}{2} \right\rangle_{\hat{q}}$$

- The Wigner function forms a quasi-probability distribution

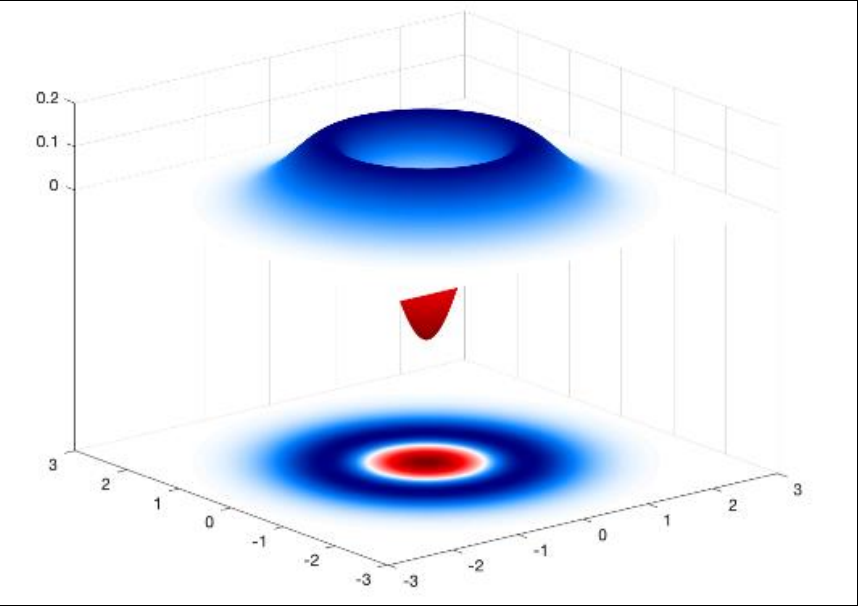


Wigner negativity

- Wigner negativity shows genuine non-Gaussianity

$$\|W_\rho^{\mathrm{CV}}\|_1 = \int \mathrm{d}\boldsymbol{r}\,\left|W_\rho^{\mathrm{CV}}(\boldsymbol{r})\right|$$

- Monotone under “Gaussian protocols”
  - Gaussian unitary
  - Attaching vacuum
  - Gaussian measurements
  - Gaussian feedforward



[Takagi, Zhuang, Phys. Rev. A ’18]

[Albarelli et al., Phys. Rev. A ’18]

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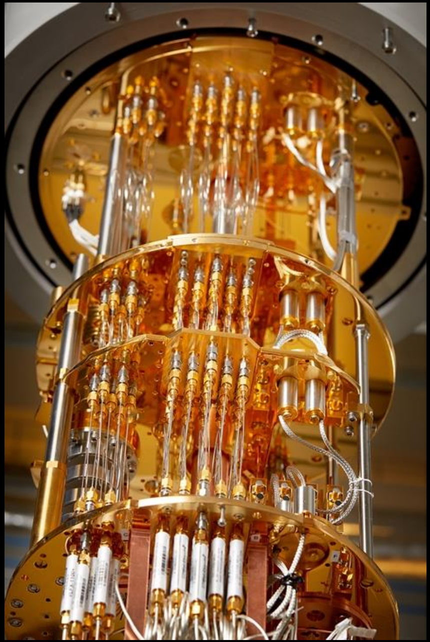
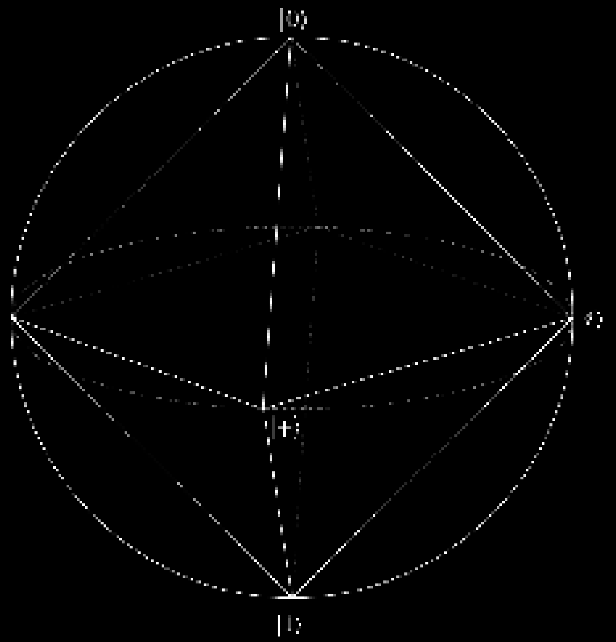
Discrete Variables

- Qudit Pauli $\hat{P}_d(\boldsymbol{u}) = \otimes_{i=1}^{n} \omega_d^{\frac{1}{2}a_i b_i} \hat{X}_d^{a_i} \hat{Z}_d^{b_i}$
- A universal gate set is $\{\underbrace{\hat{R}, \hat{P}, \mathrm{S\hat{U}M}}_{\text{Clifford}}, \hat{T}\}$
- Clifford unitaries: $\hat{U}_C \hat{P}_d(\boldsymbol{u}) \hat{U}_C^{\dagger} = \hat{P}_d(S\boldsymbol{u})$



Stabilizer states

- Pure stabilizer states are the extreme points of the octrahedron
- Closed under Clifford operations
- Eigenstates of Pauli operators



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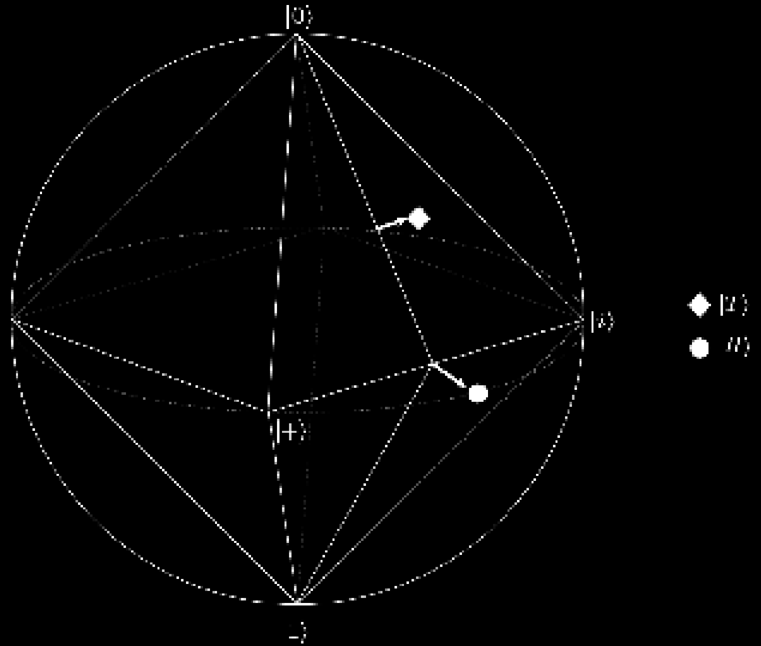
Magic states

- Magic states enable non-Clifford operations through teleportation
- Qubit: Most famous examples *T* and *H* type

$$|H\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{i\pi/4}\,|1\rangle\Big)$$

$$|T\rangle = \cos(\beta)\,|0\rangle + \sin(\beta)e^{i\pi/4}\,|1\rangle$$

$$\cos(2\beta) = \frac{1}{\sqrt{3}}$$



Why stabilizer and magic states?

Gottesman-Knill theorem

A quantum computer based only on:

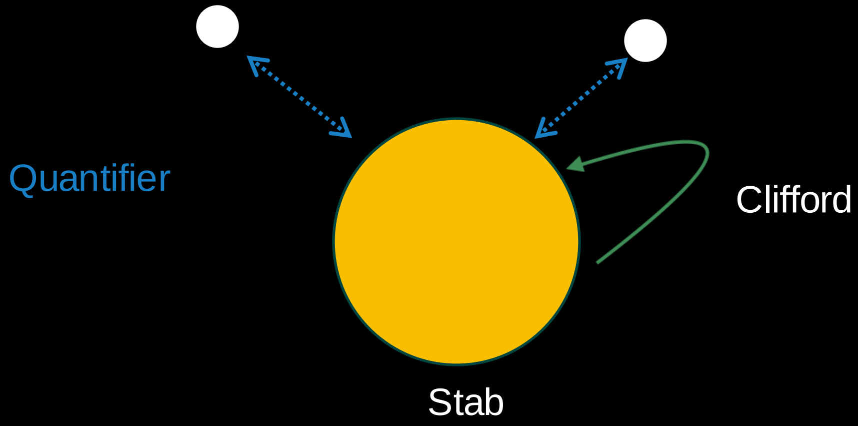
1. Qudits initialized in a Pauli eigenstate
2. Clifford group operations
3. Pauli measurements

Can be simulated efficiently with a classical computer

[Gottesman ’98]

Magic

- Stabilizer circuits are classical simulatable
- Magic is a necessary resource for quantum advantage



DV Wigner function

Odd Dimensions

- Phase-space representation of a DV quantum state $W_\rho^{\mathrm{DV}}(\boldsymbol{u}) = d^{-n}\,\mathrm{Tr}\left[\hat{A}(\boldsymbol{u})\hat{\rho}\right]$

$$\hat{A}(\boldsymbol{u}) = d^{-n} \sum_{\boldsymbol{v}\in\mathbb{Z}_d^{2n}} \omega_d^{-\boldsymbol{u}\Omega_n\boldsymbol{v}^T} \hat{P}_d(\boldsymbol{v})^\dagger$$

- Wigner negativity $\|W_\rho^{\mathrm{DV}}\|_1 = \sum_{\boldsymbol{u}} \left|W_\rho^{\mathrm{DV}}(\boldsymbol{u})\right|$

- Monotone under “Stabilizer protocols”
  - Clifford unitary
  - Auxilliary computation basis states
  - Pauli measurements
  - Clifford feedforward

[Veitch et al., New J. Phys. ’14]

Connecting CV and D

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DV

- Pauli
- Clifford
- DV Wigner function
- Magic
  - Negativity of Wigner function

CV

- Displacements
- Gaussian
- CV Wigner function
- Non-Gaussianity
  - Negativity of Wigner function

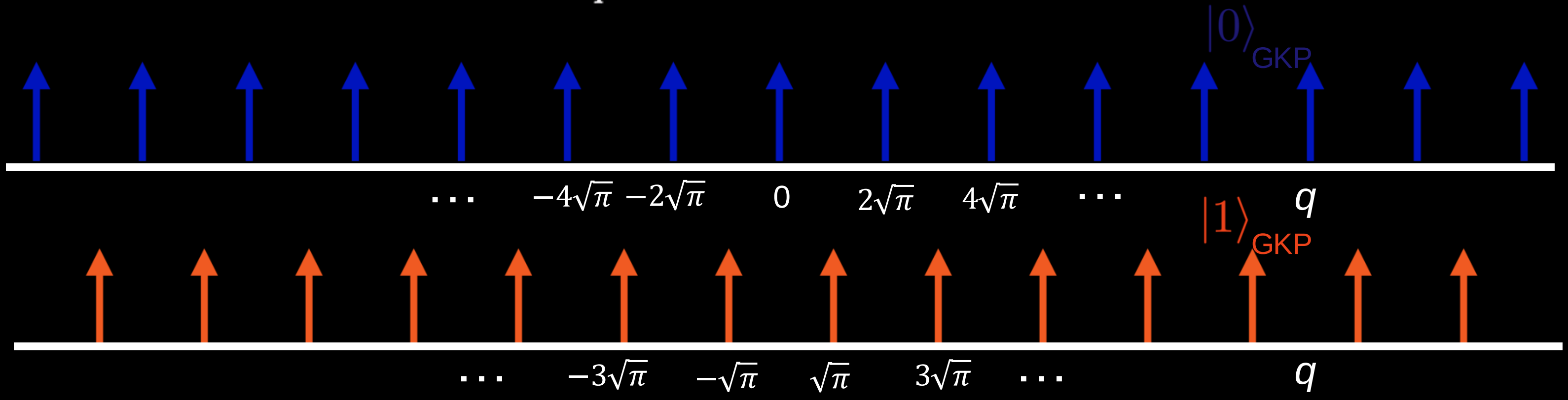
Any mapping between magic and non-Gaussian measures for a given state?

Quantitative connections between DV and CV

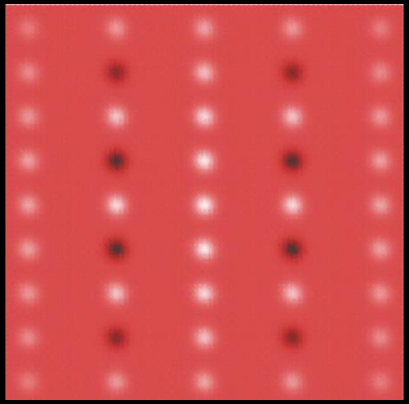
Gottesman-Kitaev-Preskill Code

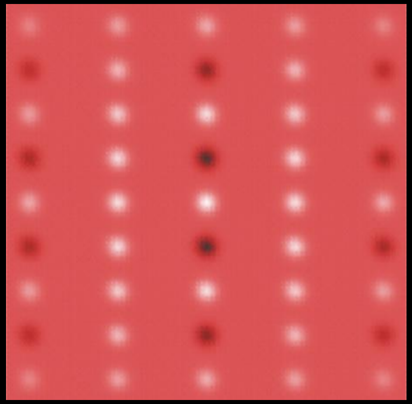
- Error correction code for continuous-variable systems that encode qudits

$$|j\rangle_{\mathrm{GKP}} = \sum_{s=-\infty}^{\infty} \left| \sqrt{\frac{2\pi}{d}}(j+ds) \right\rangle_{\hat{q}}$$



Wigner functions of GKP states



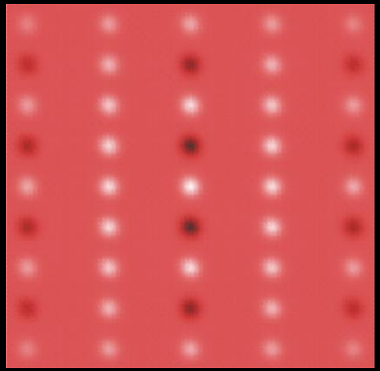


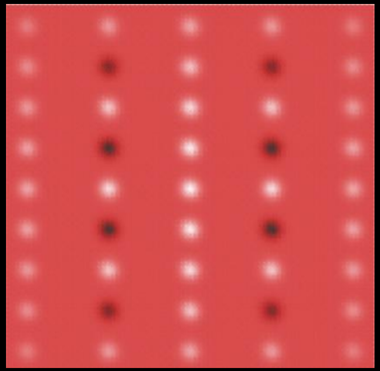
- (Ideal) GKP states are unnormalizable and have infinite non-Gaussianty
- The Wigner function is periodic with a unit cell $[0, \sqrt{2d\pi})$

Wigner functions of GKP states

- Consider negativity of one unit cell

$$\|W^{\mathrm{CV}}_{\rho_{\mathrm{GKP}}}\|_{1,\mathrm{cell}} = \int_{\mathrm{cell}} \mathrm{d}\boldsymbol{r} \left| W^{\mathrm{CV}}_{\rho_{\mathrm{GKP}}} \right|$$





- Constant for pure stabilizer states
- Is there a connection between non-Gaussianity and magic?

[Yamasaki, Matsuura, Koashi, PRR ’20]
[Hahn et al. PRL ’22]

CV-DV connection with Wigner function

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- We consider the operator basis $\hat{O}_{l,m} = \omega_d^{-ml/2} \hat{M}_l \hat{Z}_d^m$ $\quad \hat{M}_l = \sum_{x \in \mathbb{Z}_d} |l - x\rangle \langle x|$
- Coefficients of this basis $x_{\hat{\rho}}(\boldsymbol{l}, \boldsymbol{m}) = d^{-n} \operatorname{Tr}\left(\hat{O}_{\boldsymbol{l},\boldsymbol{m}} \hat{\rho}\right)$
- Wigner function

$$W^{\mathrm{CV}}_{\rho_{\mathrm{GKP}}}(\boldsymbol{r}) = \frac{\sqrt{d}^n}{\sqrt{8\pi}^n} \sum_{\boldsymbol{l},\boldsymbol{m}} c_{\rho_{\mathrm{GKP}}}(\boldsymbol{l}, \boldsymbol{m}) \delta\left(\boldsymbol{r_p} - \boldsymbol{m}\sqrt{\frac{\pi}{2d}}\right) \delta\left(\boldsymbol{r_q} - \boldsymbol{l}\sqrt{\frac{\pi}{2d}}\right)$$

$$c_{\rho_{\mathrm{GKP}}}(\boldsymbol{l}, \boldsymbol{m}) = x_\rho(\boldsymbol{l}, \boldsymbol{m})$$

CV DV

CV-DV connection with Wigner function

- General connection between Wigner negativity and magic
- For a n-qudit state we get

$$\|x_\rho\|_1 = \frac{\|W^{\mathrm{CV}}_{\rho_{\mathrm{GKP}}}\|_{1,\mathrm{cell}}}{\|W^{\mathrm{CV}}_{\mathrm{STAB}_n,\mathrm{GKP}}\|_{1,\mathrm{cell}}}$$

Magic Non-Gaussianity

- For odd: $\|x_\rho\|_1 = \|W^{\mathrm{DV}}_\rho\|_1$

Magic measure

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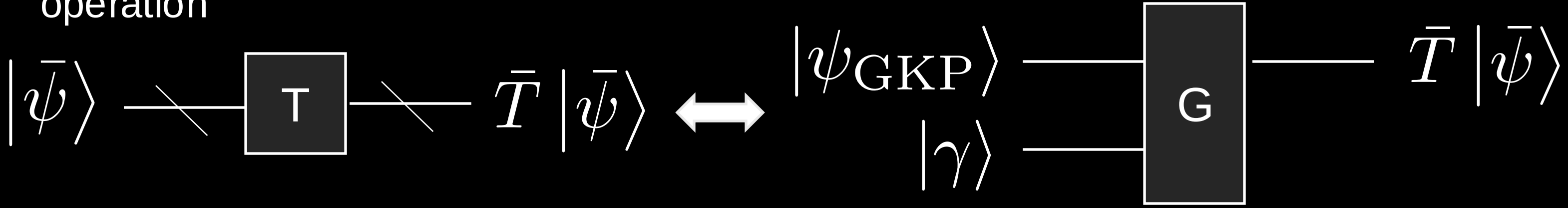
- For odd dimensions, $\left\|x_\rho\right\|_1$ coincides with Wigner negativity $\left\|W_\rho^{\mathrm{DV}}\right\|$
- For even dimensions, $\left\|x_\rho\right\|_1$ serves as a magic measure in the following sense:
  - Invariance under Clifford unitaries $\left\|x_{U_C \rho U_C^\dagger}\right\|_1 = \|x_\rho\|_1$
  - Every pure stabilizer state $\hat{\phi}$ takes the minimum value $\|x_\phi\|_1 = 1$
  - For multi-qubit systems, $\|x_\phi\|_1 = 1$ if and only if $\hat{\phi}$ is a stabilizer state

Application

- In GKP code, logical Clifford operations can be implemented by Gaussian operations.

- Known implementations for logical non-Clifford operations use non-Gaussian operation



- Natural guess: Non-Gaussian operations are needed to implement logical non-Clifford gates.

Not obvious because GKP states already have non-Gaussianity initially.

Non-Clifford needs non-Gaussianity

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Suppose Λ is a channel with n-qubit input and output. If there is a pure magic state $\hat{\psi}$ and a pure stabilizer state $\hat{\phi}$ such that $\Lambda(\hat{\phi}) = \hat{\psi}$, then Λ cannot be implemented in the GKP code space by Gaussian protocols composed by

- Feedforwarded Gaussian operations conditioned on the measurement outcomes
- Gaussian unitary
- Attaching vacuum
- Gaussian measurements

- Extending previous finding for specific qubit operations
- For odd dimensions, the condition can be relaxed to the existence of a stabilizer state $\hat{\sigma}$ and a state $\hat{\rho}$ such that $\Lambda(\hat{\sigma}) = \hat{\rho}$ and $\|W_\rho^{\mathrm{DV}}\|_1 > 1$

[Yamasaki et al. Phys. Rev. Res. '20]

Summary & Outlook

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- Established the quantitative connection between magic (DV) and non-Gaussian (CV) by GKP encoding via Wigner and characteristic functions.
  - Showed that non-Clifford gate in GKP code space cannot be implemented by a Gaussian protocol
  - Proposed a simulation algorithm based on the distribution defined by a Hermitian extension of Pauli operators
- Finite squeezing?
- Other DV-CV connections with different bosonic codes?