ENTANGLEMENT GENERATION FROM ATHERMALITY

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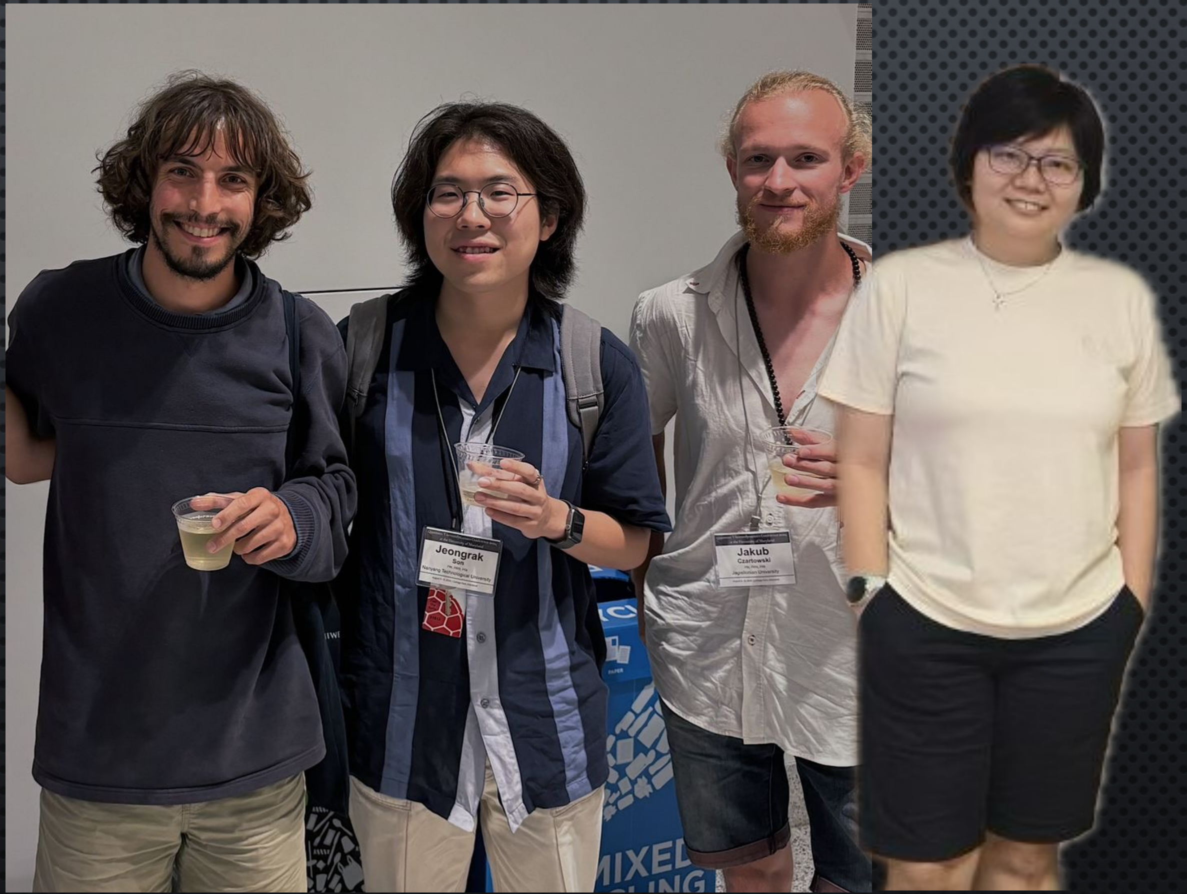


PRR **6**, 033236 (2024)

MOTIVATION

Entanglement generation in the presence of thermodynamical constraints

THERMODYNAMIC RESOURCE THEORIES

Thermal environment with a **fixed temperature**

What are NOT allowed thermodynamically?

- transforming Gibbs states into athermal states (**Gibbs preserving**)
- generating energy-coherence (**covariant**)

THERMODYNAMIC RESOURCE THEORIES

What are allowed thermodynamically?

- attaching to thermalised environments
- detaching from environments
- energy-preserving closed dynamics

Janzing et al. Int. J. Theor. Phys. **39**, 2717 (2000)

Thermal operations:

$$\rho_S \mapsto \Phi(\rho_S) = \mathrm{Tr}_E[U_{SE}(\rho_S \otimes \gamma_E^\beta)U_{SE}^\dagger]$$

- $[U_{SE}, H_S \otimes 1_E + 1_S \otimes H_E] = 0$
- $\gamma_E^\beta = \dfrac{e^{-\beta H_E}}{\mathrm{Tr}[e^{-\beta H_E}]}$

Work extraction:

Horodecki & Oppenheim Nat. Commun. **4**, 2059 (2013)

single shot, energy-incoherent: $\rho \otimes |0\rangle\langle 0| \rightarrow |1\rangle\langle 1|$

$$W = \beta^{-1} D_{\min}(\mathbf{p} || \gamma^{\beta})$$

Brandao et al. PNAS **112**, 3275 (2015)

asymptotic/catalytic: $\rho \otimes |0\rangle\langle 0| \xrightarrow{\epsilon} |1\rangle\langle 1|$

$$W = \beta^{-1} S(\rho || \gamma^{\beta})$$

Heat bath algorithmic cooling:

Alhambra, Lostaglio & Perry Quantum **3**, 188 (2019)

qubit ground state population

$$p^{(k)} \mapsto p^{(k+1)} = 1 - e^{-\beta E}(1 - p^{(k)})$$

Erasure:

Horodecki & Oppenheim Nat. Commun. **4**, 2059 (2013)

$$\gamma_S \otimes |1\rangle\langle 1|_B \to |0\rangle\langle 0|_S \otimes |0\rangle\langle 0|_B$$

if $e^{-\beta E_B} \leq Z_S^{-1}$

Can we aim for **quantum properties?**

- energy coherence
  - cannot be generated (GPC, TO)
- entanglement
  - yes, if we have enough athermality

ENTANGLEMENT GENERATION

Entanglement generation with thermodynamically free unitaries $[U,H]=0$

e.g. when $H=E|1\rangle\langle 1|$ for each qubit,

$$|{+}0\rangle \xrightarrow{\text{CNOT}} |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$: **not free**

$$|01\rangle \xrightarrow{\exp(i\frac{\pi}{4}(X\otimes X+Y\otimes Y+Z\otimes Z))} |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$: **free**

ENTANGLEMENT GENERATION

Allowed operations: $U_{\mathscr{H}} = \tilde{U}_{\mathscr{V}} \oplus 1_{\mathscr{H} \setminus \mathscr{V}}$

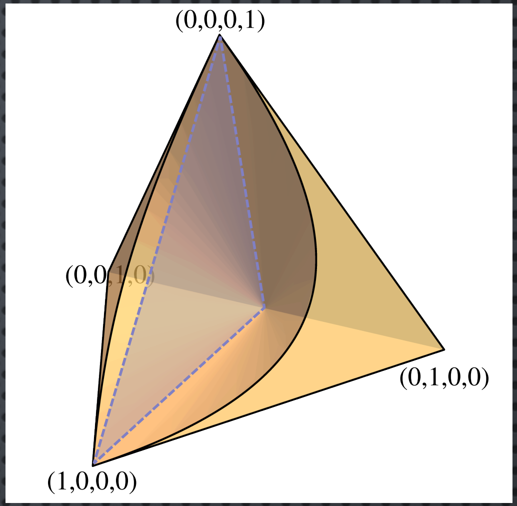
$\mathscr{V} = \mathrm{span}\{|01\rangle, |10\rangle\}$: E-degenerate subspace

$|00\rangle$, $|11\rangle$: energetically unique states

$\Rightarrow$ cannot generate entanglement without changing the energy

ENTANGLEMENT GENERATION

$\mathbb{E}$ for incoherent two qubits



Subspace (non)entanglable set

$\mathbb{E} = \{\rho \mid \min_{U \in \mathcal{U}_{\mathrm{EP}}} f(U\rho U^{\dagger}) < 0\}$: subspace entanglable

- $f(\rho)<0$ if ρ: entangled (e.g. negativity)
- $\mathcal{U}_{\mathrm{EP}}$: set of all energy-preserving unitaries

ENTANGLEMENT GENERATION

Can we improve entanglement generation with a heat bath?

- cold heat baths might make a state purer
- hot heat baths might provide energy

$\mathscr{T}_+(\rho) = \{\sigma |\ \exists \Phi \in \mathrm{TO}$, s.t. $\Phi(\rho) = \sigma\}$: set of reachable states from ρ via thermal operations

- when $[\rho, H] = 0$, thermomajorisation fully characterises $\mathscr{T}_+(\rho)$

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Thermally (non)entanglable set:

$$\mathbb{TNE} = \{\rho \,|\, \mathscr{T}_{+}(\rho) \subset \mathbb{SEP}\}$$

Thermally entanglable set:

$$\mathbb{TE} = \mathbb{D} \backslash \mathbb{TNE}$$

RESULTS

Entanglement generation can always be separated into two steps

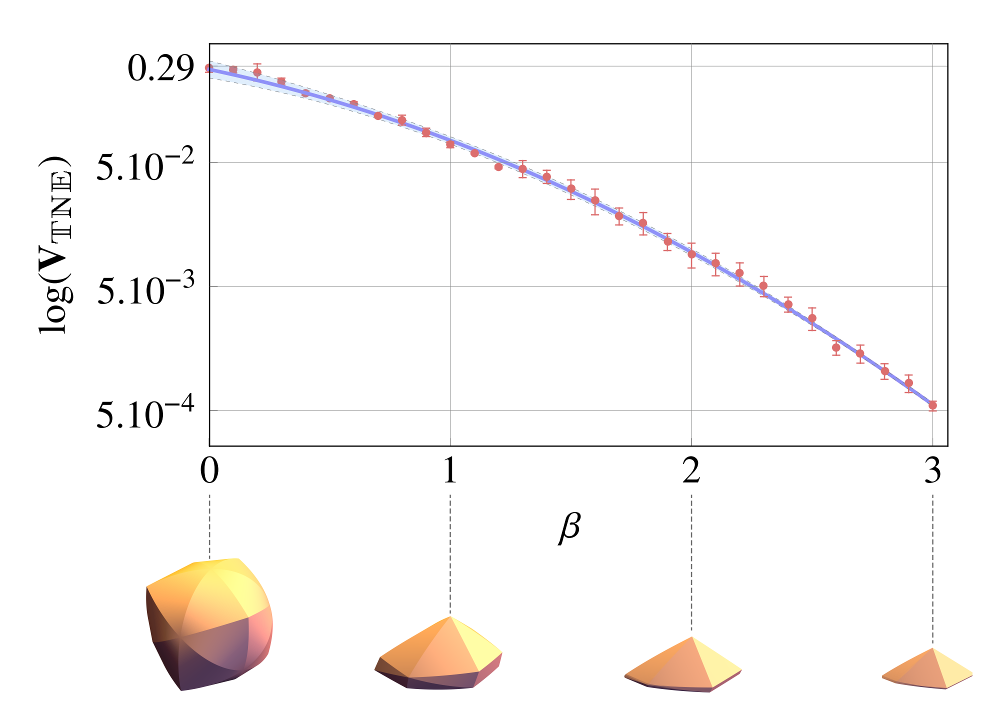
1. population transformation via thermal operations
2. unitary transformation within the E-degenerate subspace

Theorem ($\mathbb{TNE}$ for two qubits)

An energy-incoherent state ρ is thermally entanglable if and only if ρ^* is unitarily entanglable

ρ^*: extreme point of $\mathscr{T}_+(\rho)$ corresponding to a specific β-ordering

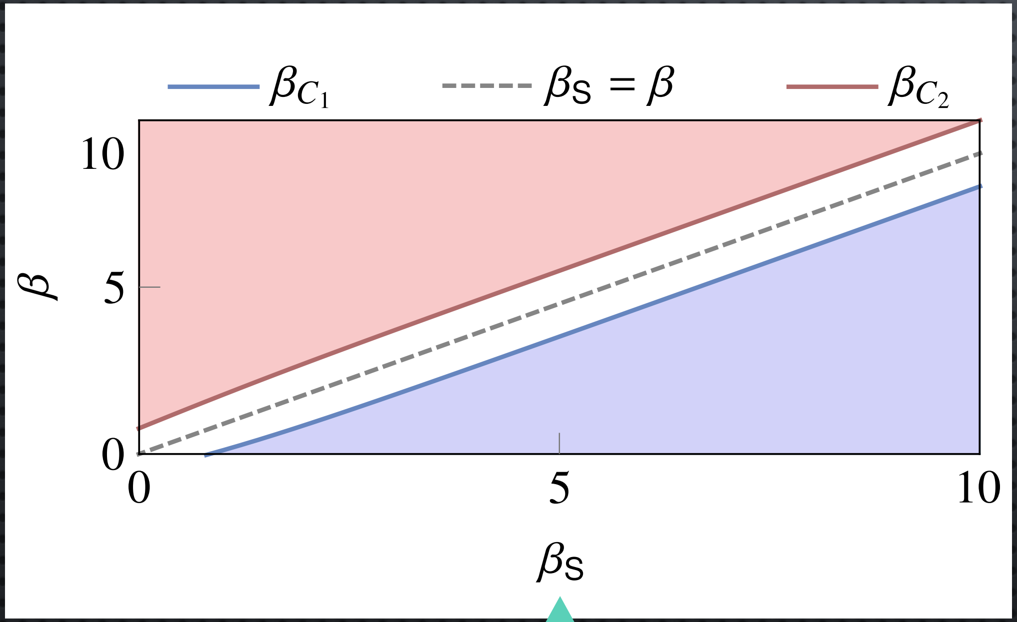
Environment temperature vs. entanglement



Lower temperature (higher β)
$\Rightarrow$ more entanglable states (smaller $\mathbb{TNE}$)

As $T \to 0$ all states (except the ground state) become entanglable

Environment temperature vs. entanglement



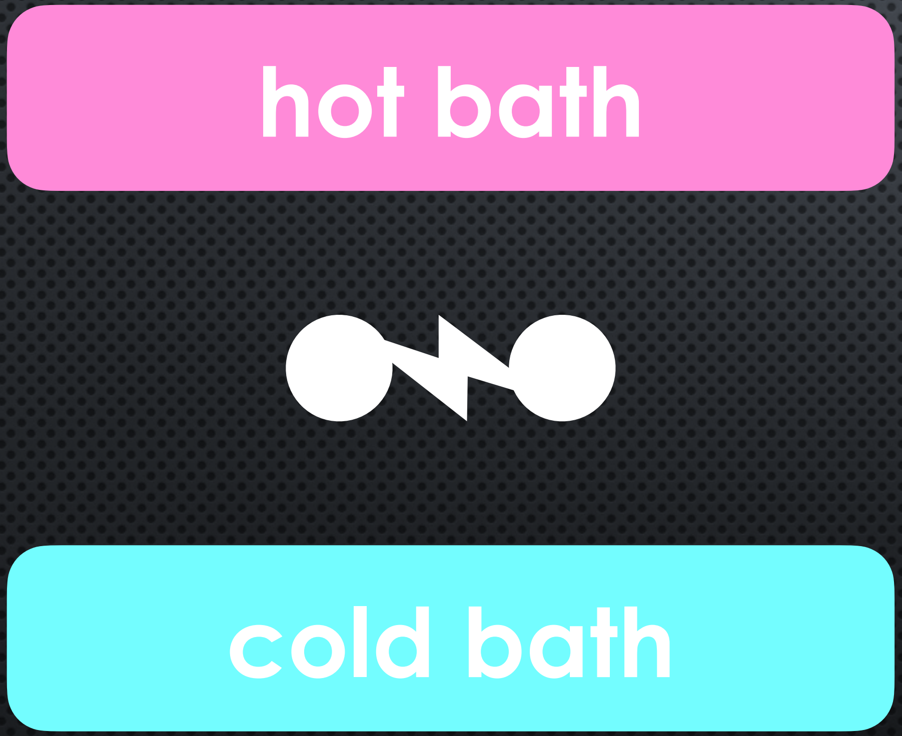
initial state inverse temperature

Assume initially thermal state:

$$|\beta - \beta_S|E \gtrsim \log 3$$

$\Rightarrow$ entanglable

Environment temperature vs. entanglement



Heat engines outputting entanglement?

1. thermalise with the hot bath
2. entangle using the cold bath
3. replace it with a fresh state

REMARKS: THERMODYNAMICALLY DISTINCT ENTANGLEMENTS

$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$: can be generated from E-incoherent states + heat bath

vs.

$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$: cannot be generated from any E-incoherent states + heat bath

REMARKS: THERMODYNAMICALLY DISTINCT ENTANGLEMENTS

Non-equilibrium free energy:

$$F(\rho) := \langle H \rangle_\rho - \beta^{-1} S(\rho) = \beta^{-1} S(\rho || \gamma^\beta) - \beta^{-1} \log Z_S$$

Bell states have the same free energy

$$F(\Psi^\pm) = F(\Phi^\pm) = E$$

$\Rightarrow |\Psi^\pm\rangle^{\otimes n} \xrightarrow{\epsilon} |\Phi^\pm\rangle^{\otimes n}$ with $n \to \infty$ (TO)

REMARKS: THERMODYNAMICALLY DISTINCT ENTANGLEMENTS

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Bell states have the same free energy

$$F(\Psi^\pm) = F(\Phi^\pm) = E$$

$\Rightarrow$asymptotically/catalytically equivalent (GP, GPC)

Shiraishi & Sagawa PRL **126**, 150502 (2021); Shiraishi arXiv:2406.06234

REMARKS: THERMODYNAMICALLY DISTINCT ENTANGLEMENTS

$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$: can be generated when $E_1 = E_2 = E_3$.

$|\widetilde{\mathrm{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle)$: can be generated when $E_1 + E_2 = E_3$.

REMARKS: THERMODYNAMICALLY DISTINCT ENTANGLEMENTS

Energy structures must be tuned so that entangled states are **symmetric**: $e^{-iHt}\rho e^{iHt} = \rho$ for all $t \in \mathbb{R}$

Is symmetric entanglement always as good as asymmetric ones?

TAKE HOME MESSAGES AND OUTLOOKS

- Entanglement generation from athermality can be fully characterised for the two-qubit case
- Entangled states can be thermodynamically distinct with the same amount of entanglement
- Need to study **quantum outputs** from quantum thermodynamic operations more