# Approximation algorithms for higher-order refinements in resource theories

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# Resource interconversion

# Channel interconversion

# Channel interconversion (fully classical)

- One-shot task: Given channels $W_{X\to Y}(y|x)$ and $\mathrm{V}_{X\to Y}(y|x)$, how well can one channel simulate the other one?
- Reverse Shannon theorem [Bennett *et al.* 02]: Optimal asymptotic rate with shared randomness assistance

$C(W \mapsto V) = \frac{C(W)}{C(V)}$ with $C(W) = \sup_{P_X} I(X:Y)_{W(P)}$ mutual information

- Shared randomness for reversibility – higher-order refinements unknown

→ simpler tasks: channel coding $(V = \mathrm{id})$ & channel simulation $(W = \mathrm{id})$

# Channel coding

# Channel coding

- One-shot task: given channel $W_{X\to Y}(y|x)$ and message size $M$, compute success probability

$$\mathrm{p}(W \mapsto \mathrm{id}_M) := \sup_{(E,D)} \frac{1}{M} \sum_{x,y,i} W_{X\to Y}(y|x)E(x|i)D(i|y)$$

  over encoder-decoder pairs

- Shannon theorem: Largest $r$ with $\mathrm{p}(W^{\times n} \mapsto \mathrm{id}_{2^{rn}}) \to 1$ in limit $n \to \infty$ is quantified by channel capacity $C(W)$ with mutual information formula

$$C(W) = \sup_{P_X} I(X:Y)_{W(P)}$$

- Higher-order refinements?

# Meta converse for channel coding

- Bottom-up approach to Shannon theory: Meta converse linear program relaxation [Hayashi 09, Polyanskiy *et al.* 10]

$$\mathrm{p}(\mathrm{W} \mapsto \mathrm{id}_M) \leq \mathrm{p}_{\mathrm{ns}}(W \mapsto \mathrm{id}_M) \coloneqq \sup_{(r,p)} \frac{1}{M} \sum_{x,y} W_{X \to Y}(y|x) r(x,y)$$

with $\sum_{x,y} r(x,y) \leq 1, \sum_x p_x = M, r(x,y) \leq p_x, 0 \leq r(x,y), p(x) \leq 1$

- Physics: Corresponds to non-signaling assisted value [Matthews 11]
- Bottom-up approach to Shannon theory: one-shot optimality?

# Optimality of meta converse?

- Yes, rounding methods from approximation algorithms give for $M, N$ that [Barman & Fawzi 15]

$$\mathrm{p_{ns}}(W \mapsto \mathrm{id}_N) \geq \mathrm{p}(W \mapsto \mathrm{id}_N) \geq \frac{M}{N} \cdot \left(1 - \left(1 - \frac{1}{M}\right)^N\right) \cdot \mathrm{p_{ns}}(W \mapsto \mathrm{id}_M)$$

which implies the constant factor approximation

$$\mathrm{p_{ns}}(W \mapsto \mathrm{id}_M) \geq \mathrm{p}(W \mapsto \mathrm{id}_M) \geq \frac{1}{1 - \frac{1}{e}} \cdot \mathrm{p_{ns}}(W \mapsto \mathrm{id}_M)$$

- Bound is exactly tight
- Gives strong upper bound on entanglement assistance

# Higher-order refinements

- Large deviation: For $r \geq C(V)$ strong converse exponent [Dueck & Körner 79]

$$\mathrm{SCExp}(\mathrm{V} \mapsto \mathrm{id}, \mathrm{r}) := \lim_{n\to\infty} -\frac{1}{n}\log \mathrm{p}(V^{\times n}, \mathrm{id}_{2^{nr}})$$

$$\mathrm{SCExp}(\mathrm{V} \mapsto \mathrm{id}, \mathrm{r}) = \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\cdot(r - C_\alpha) \text{ with } \mathrm{C}_\alpha \text{ Rényi capacities}$$

- Large deviation: For $\mathrm{r_c} \leq r \leq C(V)$ error exponent [Shannon *et al.* 67]

$$\mathrm{Exp}(\mathrm{V} \mapsto \mathrm{id}, \mathrm{r}) := \lim_{n\to\infty} -\frac{1}{n}\log(1-\mathrm{p}(V^{\times n} \mapsto \mathrm{id}_{2^{nr}}))$$

$$\mathrm{Exp}(\mathrm{V} \mapsto \mathrm{id}, \mathrm{r}) = \sup_{\alpha\in(0,1]}\frac{1-\alpha}{\alpha}\cdot(C_\alpha(V) - r)$$

- Small and moderate deviation [Hayashi 09 , Polyanskiy et al. 10]

# Channel simulation

# Channel simulation

- One-shot task: given channel $V_{X\to Y}(y|x)$ and identity channel of size $M$ compute, compute success probability

$$\mathrm{p_{sr}}(\mathrm{id}_M \mapsto V) := \sup_{(p,E,D)} 1 - \sup_x ||\tilde{V}_{p,X\to Y}(\cdot\,|x) - V_{X\to Y}(\cdot\,|x)||_{\mathrm{TV}}$$

over synthesized channels $\tilde{V}_{p,X\to Y}(y|x) := \sum_s p(s) \sum_i E_s(i|x) D_s(y|i)$

with randomness assisted encoder-decoder pairs

- Reverse Shannon theorem: Smallest $r$ with $\mathrm{p_{sr}}\left(\mathrm{id}_{2^n} \mapsto V^{\times(nr)}\right) \to 1$ in limit $n \to \infty$ is quantified by channel capacity [Bennett *et al.* 02]

$$C(V) = \sup_{P_X} I(X\!:\!Y)_{V(P)}$$

# Meta converse for channel simulation

- Bottom-up approach for Shannon theory: Natural meta converse linear program relaxation

$$\mathrm{p}_{\mathrm{sr}}(\mathrm{id}_M \mapsto V) \leq \mathrm{p}_{\mathrm{ns}}(\mathrm{id}_M \mapsto V) := \sup_{(U,q)} 1 - \sup_x ||U_{X\to Y}(\cdot\,|x) - V_{X\to Y}(\cdot\,|x)||_{\mathrm{TV}}$$

over channels $U_{X\to Y}(y|x)$ with $U_{X\to Y}(y|x) \leq q(y)$ and $\sum_y q(y) = M$

- Physics: corresponds to non-signaling assisted value [Fang *et al.* 20 (B.)]
- Bottom-up approach to Shannon theory: One-shot optimality?

# Result: Optimality of meta converse

- Yes, rounding methods from approximation algorithms give for $M, N$ that

$$\mathrm{p_{ns}}(\mathrm{id}_N \mapsto V) \geq \mathrm{p_{sr}}(\mathrm{id}_N \mapsto V) \geq 1 \cdot \left(1 - \left(1 - \tfrac{1}{M}\right)^N\right) \cdot \mathrm{p_{ns}}(\mathrm{id}_M \mapsto V)$$

which implies the constant factor approximation

$$\mathrm{p_{ns}}(\mathrm{id}_M \mapsto V) \geq \mathrm{p_{sr}}(\mathrm{id}_M \mapsto V) \geq \frac{1}{1-\frac{1}{e}} \cdot \mathrm{p_{ns}}(\mathrm{id}_M \mapsto V)$$

- Bound is exactly tight
- Gives strong upper bound on entanglement assistance

# Result: Higher-order refinements

- Large deviation: For $r \geq 0$ strong converse exponent

$$\mathrm{SCExp_{sr}}(\mathrm{id} \mapsto \mathrm{V}, \mathrm{r}) \coloneqq \lim_{n\to\infty} -\frac{1}{n} \log \mathrm{p_{sr}}\left(\mathrm{id}_{2^n} \mapsto \mathrm{V}^{\times(nr)}\right)$$

$$\mathrm{SCExp_{sr}}(\mathrm{id} \mapsto \mathrm{V}, \mathrm{r}) = \sup_{\alpha\in[0,1]} (1-\alpha)\cdot(C_\alpha(V) - r)$$

- Large deviation: For $r \geq 0$ error exponent

$$\mathrm{Exp_{sr}}(\mathrm{id} \mapsto \mathrm{V}, \mathrm{r}) \coloneqq \lim_{n\to\infty} -\frac{1}{n} \log(1 - \mathrm{p_{sim}^{sr}}(V^{\times n}, 2^{nr}))$$

$$\mathrm{Exp_{sr}}(\mathrm{id} \mapsto \mathrm{V}, \mathrm{r}) = \sup_{\alpha\geq 0} \alpha\cdot(r - C_{\alpha+1}(V))$$

- Small and moderate deviation [Cao et al. 24 (B.)]

# Proof ideas

# Proof: Tightness of gap

- The family of channels

$$\mathrm{U}^{(n,k)} : \binom{n}{k} \to \{1, 2, \cdots, n\} \text{ with } U_{X\to Y}(y|x) := \frac{1}{k}\, 1\{y \in x\}$$

has for $n = M^2$ and $k = M$ the limit

$$\lim_{M\to\infty} \frac{\mathrm{p_{sr}}\left(\mathrm{id}_M \mapsto U^{(M^2,M)}\right)}{\mathrm{p_{ns}}\left(\mathrm{id_M} \mapsto U^{(M^2,M)}\right)} = 1 - \frac{1}{e}$$

which then exactly matches

$$\mathrm{p_{ns}}(\mathrm{id}_M \mapsto V) \geq \mathrm{p_{sr}}(\mathrm{id}_M \mapsto V) \geq \frac{1}{1-\frac{1}{e}} \cdot \mathrm{p_{ns}}(\mathrm{id}_M \mapsto V)$$

- Crucial step: upper bound power of shared randomness assistance

# Proof: Rounding

- Any feasible solution of linear program $\mathrm{p_{ns}}(V, M)$ gives channel $U_{X\to Y}(y|x)$ and distribution $\frac{1}{M}q(y)$

  → construct shared randomness assisted synthesized channel $\tilde{V}_{p,X\to Y}$ for $N$ such that

$$U_{X\to Y}(y|x) \geq \left(1-\left(1-\frac{1}{M}\right)^N\right)\tilde{V}_{p,X\to Y}(y|x) \quad \forall\, x, y$$

- Basic idea:
  1. Shared randomness assistance $\{\frac{1}{M}q(y)\}_y$
  2. Rejection sampling with $N$ steps, $t_{\mathrm{initial}}(y) = \frac{1}{M}q(y)$, $t_{\mathrm{target}}(y) = U_{X\to Y}(y|x)$
- Above meta inequality works for any average error / fidelity criteria

# Channel interconversion

# Result: Strong converse exponent interconversion

- From $W_{X\to Y}(y|x)$ to $\mathrm{V}_{\mathrm{X}\to Y}(y|x)$ via shared randomness-assisted synthesized channels

$\tilde{V}_{p,X\to Y}(y|x) \coloneqq \sum_s p(s) \sum_i E_s(i|x) W(x|y) D_s(y|i)$ in variational distance

$$\mathrm{p_{sr}}(W \mapsto V) \coloneqq \sup_{(p,E,D)} 1 - \sup_x ||\tilde{V}_{p,X\to Y}(\cdot\,|x) - V_{X\to Y}(\cdot\,|x)||_{\mathrm{TV}}$$

or in fidelity

$$\mathrm{F_{sr}}(W \mapsto V) \coloneqq \sup_{(p,E,D)} \inf_x F\left(\tilde{V}_{p,X\to Y}(\cdot\,|x) - V_{X\to Y}(\cdot\,|x)\right)$$

- Large deviation: For $r \geq C(W \mapsto V)$ strong converse exponent in fidelity

$$\mathrm{SCExp_F}(W \mapsto V, r) \coloneqq \lim_{n\to\infty} -\frac{1}{n} \log \mathrm{F_{sr}}\left(W^{\times n} \mapsto V^{\times(rn)}\right)$$

$$\mathrm{SCExp_F}(W \mapsto V, r) \coloneqq \sup_{\frac{1}{2}\leq\alpha\leq 1} \frac{1-\alpha}{\alpha}\left(r \cdot C_\alpha(V) - C_{\frac{\alpha}{2\alpha-1}}(W)\right)$$

# Conclusion

- Previous channel coding result

$$\mathrm{p_{ns}}(W \mapsto \mathrm{id}_N) \geq \mathrm{p}(W \mapsto \mathrm{id}_N) \geq \frac{M}{N} \cdot \left(1 - \left(1 - \frac{1}{M}\right)^N\right) \cdot \mathrm{p_{ns}}(W \mapsto \mathrm{id}_M)$$

via maximizing of sub-modular function rounding, tight → exponents ✓

- Novel channel simulation result

$$\mathrm{p_{ns}}(\mathrm{id}_N \mapsto V) \geq \mathrm{p_{sr}}(\mathrm{id}_N \mapsto V) \geq 1 \cdot \left(1 - \left(1 - \frac{1}{M}\right)^N\right) \cdot \mathrm{p_{ns}}(\mathrm{id}_M \mapsto V)$$

via rejection sampling rounding, tight → exponents ✓

- Channel interconversion: Strong converse exponent ✓, but one-shot bounds

$$\mathrm{p_{sr}}(W \mapsto V) \text{ versus } \mathrm{p_{ns}}(W \mapsto V)?$$

# Outlook

- Other higher-order refinements for channel interconversion?
- Quantum extensions, entanglement-assistance?
  → some but not all, check out references!

- For other (quantum) resource theories: Approximation algorithms for tight one-shot bounds + higher-order refinements?

- Postdoc and PhD positions at RWTH Aachen University
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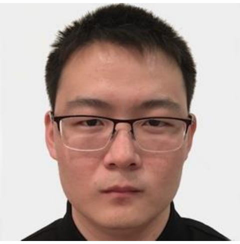
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# Quantum extensions

# Classical-quantum channels

- Classical input – quantum output channels

$$V_{X\to B}: x \mapsto \rho_B^x$$

  for sub-normalized quantum states $\rho_B^x \geq 0$, normalized to $\sum_x \mathrm{Tr}[\rho_B^x] = 1$

- Entanglement-assisted simulation protocol

$$\mathrm{p_{ea}}(\mathrm{id_M} \mapsto V) := \sup_{(\sigma,E,D)} 1 - \sup_x ||\tilde{V}_{\sigma,X\to B} - V_{X\to B}||_1$$

  over synthesized channels $\tilde{V}_{\sigma,X\to B} := \sum_i N^i_{K\to B}\left(\mathrm{Tr}_{K'}\left[\left(E_x^i \otimes 1_{K'}\right)\sigma_{KK'}\right]\right)$

  with assistance $\sigma_{KK'}$ + encoders $\{E_x^i\}_i$ and decoders $\{N^i_{K\to B}\}_i$

- Same rounding results – not know for reverse task of channel coding!

# Fully quantum channels

- No tight, dimension-independent one-shot rounding
- Result for coding & simulation: One-shot dimension-dependent rounding between entanglement-assisted

$$\mathrm{p}_{\mathrm{ea}}(\mathrm{id}_{\mathrm{M}} \mapsto V)$$

  and non-signaling assisted success probability

$$\mathrm{p}_{\mathrm{ns}}(\mathrm{id}_{\mathrm{M}} \mapsto V)$$

- Result for coding & simulation: Tight for strong converse exponent in fidelity ✔, see also [Li & Yao 22]
- Interconversion: Higher-order refinements completely open